ANALYSIS OF A TWO-DIMENSIONAL NEGATIVE FREQUENCY MODEL IN THE CONTEXT OF PHASOR THEORY

W. Abbas and M.A. Saqib
Department of Electrical Engineering, University of Engineering and Technology, Lahore 54890, Pakistan.
Emails: uet04109@yahoo.com, saqib@uet.edu.pk

ABSTRACT

This paper explores various issues of a counterintuitive negative frequency phenomenon. A generic two dimensional frequency model has been presented to signify this mysterious mathematical consequence of complex analysis. The phasor treatment of the sinusoidal signals offers the profound mathematical tools for their effective explanation, subsequently devising this mystifying concept. So the real essence of negative frequency is explored in the context of phasor analysis. Issues related to its physical reality and existence along with its significance in modeling real systems is also discussed. An approach is developed to characterize different orientations relative to this negative frequency notion investigating the modern status of our mathematical constructs and schemes.

1. INTRODUCTION

An effective representation that can facilitate and assist the mathematical manipulations of a system and is also consistent with its conceptual roots is quite influential in characterizing that system. So a large number of approaches and tools have been developed that focus on various features of a system's performance. Multi dimensional mapping scheme has proved to be quite significant in outlining the system's attributes. Any explanation for a particular phenomenon is valid if it is rational, coherent with the actual observable facts, and also self consistent.

The development of Fourier series and Fourier Transform is an excellent example of this progressive design in which we represent a signal in terms of various combinations of basic sinusoidal signal. The Fourier Transform and the frequency spectrum are powerful tools for analyzing and measuring signals [1]. For example, we can effectively acquire time-domain signals, measure the frequency content, and convert the results to real-world units and displays.

So an effective illustration and representation of a system is inevitable for the comprehensive perception of a certain phenomenon. Thus, for an efficient treatment of the basic sinusoidal signals that serve as eigen functions of linear systems, phasor analysis in association with the complex numbers has been developed. The neat thing about a sine wave such as $v(t)$ = A $\sin(\omega t + \delta)$ is that it can be considered to be directly related to a vector of length A revolving in a circle with an angular velocity ω – in fact just the y component of the vector.

The negative frequency concept is a consequence of the phasor treatment of sinusoidal signals. Negative frequencies seem counterintuitive as in time domain we tend to perceive everything from its physical perspective but this constraint is not valid when we are dealing with our mathematical generalizations. Perhaps it is this violation of physical embodiment that has enabled us to develop enormous manifestations and versions of the same physical phenomenon. Most of the physics (but not all) can be expressed accurately in terms of math, but not all the math has "comprehensible" physical interpretations. Negative frequency is one of such challenging issues.

2. PHASOR ANALYSIS FOR THE CHARACTERIZATION OF SINUSOIDAL SIGNALS

Complex analysis with its complex plane serves as a tool in simplifying the coordination between algebra and geometry of signals [2]. The domain of real numbers is limited to one dimension but the provision of including a direction (phase) in complex mathematics elevates their significance. This is why we can call them a 2D numbers. A complex number specifies its projections on real and imaginary axes either by a certain coefficient or through equations. Complex numbers may be called as existent invisible numbers as opposed to non-existent imaginary numbers.

A vector starting from the origin of complex plane to another point in that plane is a phasor. A phasor represents the amplitude and phase of a sinusoidal function and may have many forms. Probing deep into this phasor demonstration reveals its excellent potential and ease in explaining the sine and cosine functions. Euler's identity, given below, is the greatest contributor in this context:

$$e^{jx} = \cos x + j \sin x \quad (1)$$

This formula can be interpreted as saying that the function ejx traces out the unit circle in the complex number plane as x ranges through the real numbers. Here, x is the angle that a line connecting the origin with a point on the unit circle makes with the positive real axis, measured counter clockwise and in radians. This complex exponential contains within it the whole range of sine and cosine functions that are being depicted on the imaginary and real axis respectively. So what we have in the real plane is the line segment whose length is cos x while the complex plane's line segment takes the values of $\sin x$ [Figure 1]. So with this complex exponential, sinusoidal signals have been transformed simply into vectors that exclusively characterize them.

This excellent relationship may be considered as the most influential equation of mathematics. It is actually a special case of a broader relation that links two entirely different branches of mathematics; geometry which is the study of space and algebra that focuses on the structure and quantity.

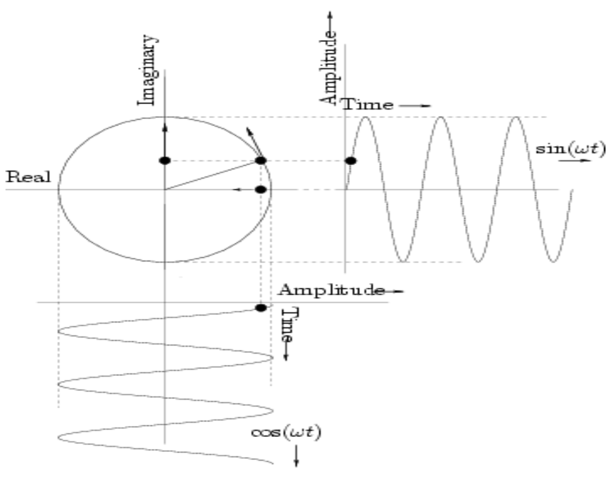


Figure 1. Relation of circular motion to sinusoidal motion via Euler's identity.

Now we will see the sine and cosine functions with a different perspective and in the context of this phasor treatment that will subsequently lead us to the real concept of negative frequency. Euler's identity provides an interpretation of the sine and cosine functions as weighted sums of the exponential function. This consequence of Euler's relation allows us to represent cosine function as a sum of two phasors; one revolving in the clockwise direction (negative) and the other in the counter clockwise direction (positive). Both of these phasors contribute equally to the result. This can be efficiently seen from the geometry as shown in Figure 2. Similarly we can demonstrate for the sine function. The presence of 'j' in the sine function only means that it is -90^{o} to the other term and hence we can mentally neglect it for a better understanding.

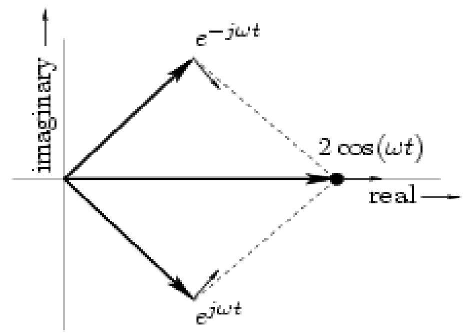


Figure 2. Opposite circular motions add to give real sinusoidal motion.

The phasors are plotted with time dimension suppressed, so they look like vector frozen in time with its plane rotating with the angular frequency. But actually the addition of a time dimension creates a corkscrew pattern [Figure 3]. The function thus looks like a helix moving forward in time to the right [1]. The x-z projection and y-z projection if plotted would give us the sine and cosine functions. So we have:

$$x = \cos \omega t \quad (2)$$

$$y = \sin \omega t \quad (3)$$

$$z = t \quad (4)$$

Here t is a real parameter. As t increases, the point (x,y,z) traces a right-handed helix of pitch 2π about the z-axis.

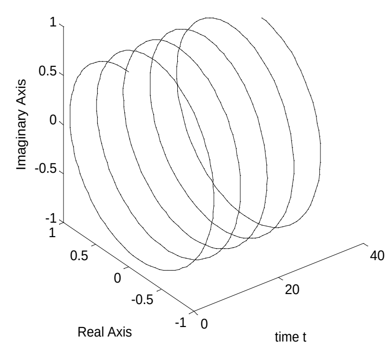


Figure 3. Three-dimensional view of the phasor $e^{j\omega t}$.

2.1. MAIN POINTS

These facts can be summarized into following main points:

- Euler's identity says that a complex sinusoid corresponds to a circular motion in the complex plane, and is the vector sum of two sinusoidal motions.

- Adding two complex conjugate waveforms together cancels out the imaginary parts and doubles the real part of each, thus giving a cosine.
- Sine wave is the difference of same two phasors divided by 2j.
- Since any real periodic signal can be represented as a sum of sine and cosine terms, thus it can also be represented by positive and negative phasors.
- As we could create a spectrum out of the coefficients of sinusoids, we can do the same thing out of the coefficients of phasors.
- For the signal to be real, every positive frequency complex sinusoid must be summed with a negative frequency sinusoid of equal amplitude. In other words, any counterclockwise circular motion must be matched by an equal and opposite clockwise circular motion in order that the imaginary parts always cancel to yield a real signal

2.2. SIGNIFICANCE OF THE PHASOR TREATMENT

The complex exponential provides a useful technique for conceptualizing the sinusoidally oscillating electrical signals. Its importance also comes from its tendency to develop a basic view for the understanding of periodic signals and to characterize the linear time-invariant signals [3]-[4].

From the Euler's identity we can easily break the signal down into its real and imaginary components. Also we can see how exponentials can be combined to represent a real signal. By modifying their frequency and phase, we can represent any signal through a superposity of many signals [5] - all capable of being represented by an exponential.

With this, a very convenient way to graph general complex frequency domain quantities at one instant of time is developed. The positions of the vectors on the + frequency axes are the frequencies of rotation. This scheme allows the presentation of either real and imaginary parts (or amplitude and phase values) on a single plot. So the inflexibility associated with the arithmetic of trigonometric identities is now confined to vector addition and subtraction problems.

3. NEGATIVE FREQUENCY

The exponential form of the Fourier series allows for negative frequency components. To this effect, the exponential series is often known as the "Bi-Sided Fourier Series", because the spectrum has both a positive and negative side [6]. Actually this concept is the outcome of the trade off between the two approaches or the tools: either a "physically correct" view that requires hard work but does not induce any controversial issues, or one that is easy to work with but comes at the expense of possible confusion over the "negative frequency" concept.

The negative frequency is there to make sure that the signal itself is real valued. The reason for negative frequencies being there is the Euler's equation:

$$\sin \omega t = (e^{j\omega t} - e^{-j\omega t}) \big/ 2j \qquad (5)$$

$$\cos \omega t = (e^{j\omega t} + e^{-j\omega t}) \big/ 2 \qquad (6)$$

Note that the concept of "negative frequency" comes from the exponents of the second terms, -jωt, that contain three factors, namely j, ω and *t*. Exactly which one of these contains the "-" sign, is a matter of choice. If we keep j = sqrt (-1) out of it, physics says that t is always increasing. Then, the only physical parameter left where the "-" sign can be "hooked on", is ω. For base band time signals, speaking purely in time domain, this is a purely mathematical construct.

Positive and negative frequencies have nothing to do with causality and the direction of time; rather they simply have to do with the direction of rotation [1]. Counterclockwise rotation is a positive angular frequency and clockwise rotation is a negative angular frequency. This terminology is confusing as in complex domain we are not talking about the frequency but explicitly the exponent of the exponential. The Q+ phasor represents the positive frequency content and the Q- phasor represents the negative frequency content because of the sign of the exponent.

Parallel to this is another outlook that it is possible to express Fourier transforms with positive frequencies only using trigonometric functions [6]. So we can say that exponential functions are easier to work with, but they require negative frequencies to do the same job. The choice of $e^{j\omega t} + e^{-j\omega t}$ with ω both negative and positive instead of sin(ω*t*) + cos(ω*t*) with only positive ω is made for convenience. The formulations are mathematically equivalent. So their existence depends on domain under observation.

3.1. A TWO - DIMENSIONAL APPROACH

Frequency is really a two dimensional concept and it is always related to a physical quality of a wave as a single dimension. Spectrum analyzers and other electrical measuring instruments are also single dimensional and thus limit our insight on this concept. The general concept of frequency can be written as:

$$f = d\varphi / dt \qquad (7)$$

where φ is the phase of a sinusoidal signal. Thus, we can define frequency as the rate of change of phase over time. So a 2π rotation over half a second means that the frequency is 2 Hz (cycles/sec.) and if the phase rotates counter clockwise it is positive: the frequency is negative if the rotation is clockwise. A much simpler and intuitive reality of this notion can be viewed with reference to time domain justification of a sine wave. If at some vertical axis we are examining the sine function signal, two parameters can entirely explain this situation: The magnitude telling us how fast the oscillations are being performed. It corresponds to the first dimension of our two-dimensional approach.

The direction of oscillation seen easily by the pattern being represented on the vertical axis i.e., either upward or downward [Figure 4]. One of them corresponding to positive frequency and the other one to the negative frequency. This direction corresponds to second dimension of our model.

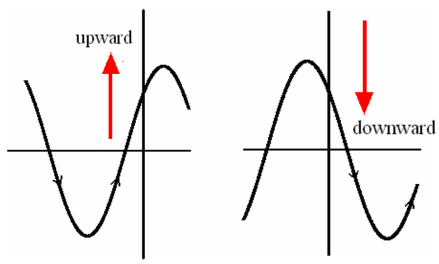


Figure 4. Change of direction in sine wave.

This all can be incorporated into the complex numbers effectively that consequently give the precise mathematical picture of negative frequency. If the trivial definition of frequency is taken as to how often something repeats - then the answer is certainly a simple positive number. But if we expand our viewpoint to two dimensional things, a natural extension is to say not only how often something repeats but, we can now include a direction. While this is a natural philosophical idea, it is motivated by mathematics. The primary reason complex numbers (two dimensional) were developed, was that one dimensional numbers proved inadequate for large classes of problems. Complex signals are nothing more than two different and separate real signals: one labeled "real" while the other labeled "imaginary" but which must be handled throughout the processing as a pair and operated upon using the rules of complex arithmetic [7].

3.2. THE PHYSICAL EXISTENCE

Again we have to convince ourselves that mathematical modeling of a system that is resourcefully enunciating has many byproduct entities. These offshoots may seem to be counter intuitive at a first glance. But since our mathematical model is an accurate one, so some physical phenomenon can be associated with its derivatives. Although this reality may not be simple and clear, yet it can be acquired through some clever reasoning. Similarly the negative frequency notion is the result of our phasor treatment and can be justified in many ways by focusing more at various issues at different times.

The negative frequency may very well be a mathematical abstraction, but with a measurable appearance. In AM modulated systems the signal is represented as two sidebands "mirrored" around the carrier frequency. The upper side band comes from modulating the baseband signal [4]. The lower side band is due to the negative frequencies. To save bandwidth one sideband can be removed, in which case we have a single-sideband (SSB) modulation scheme. The tendency of this modulating scheme to remove the negative frequency spectrum tells us about the existence of negative frequency.

When sampling real-valued signals, the negative frequencies are repeated in the band between Fs/2 and Fs (Fs is the sampling frequency). The existence of these frequency components are the sole cause of Nyquist's sampling theorem [3], that restricts the bandwidth of the signal to be sampled to f <Fs/2.

This representation is an example of elegant computational scheme. If spectrum analysis (positive frequencies only) is performed at baseband and then of the modulated signal, we find that the bandwidth of the modulated signal is twice the bandwidth of the baseband signal thus specifying the role of negative frequency components. So it is appreciative to conclude that negative frequencies exist.

The distinction between circular and cycloid filters based on the projective properties also gives a clear notion of negative frequency phenomenon. By rejecting negative frequencies circular filters produce circular paths while negative frequency infiltration in cycloid filters generates cycloid curves at the output [5].

For an angular frequency $\omega = 2\pi f$, where ω is in radians/second and f is in Hz, ω can either be a positive or negative quantity and the physical reality of this is easily seen when we explicitly write out the real and imaginary parts. For example, positive frequency [rotates counter clockwise]:

$$e^{j\omega} = e^{j2\pi f} = \cos\omega + j\sin\omega \qquad (8)$$

Real part carries the signal "cos (ωt)" and imaginary part carries the signal "sin (ωt)".

With sufficiently low frequency signals, using an oscilloscope with separate x and y-axes inputs, which can be done with a complex analog signal processor, one can display the real part on the horizontal axis and the imaginary part on the vertical axis and actually see the dot tracing out a circle in the counter-clockwise direction [positive frequency]. Negative frequency [rotates clockwise]:

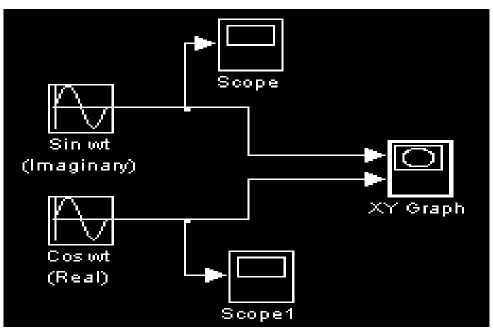
$$e^{-j\omega} = e^{-2\pi f} = \cos\omega - j\sin\omega \qquad (9)$$

Again using an oscilloscope with separate x and y inputs displaying the real and imaginary parts we can see the dot tracing out the circle in the opposite direction [clockwise or negative frequency].

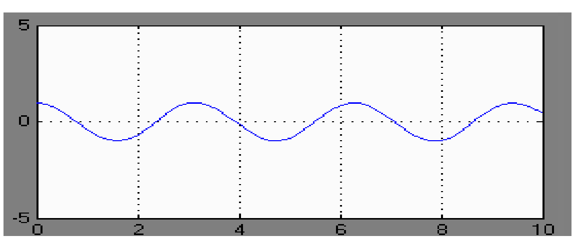
Systems supporting positive and negative frequencies must simultaneously support complex signals together with their “real” and “imaginary” signals [2]. Simulating such a system where we have separate sources for real and imaginary part of complex signal, gives us an insight on the physical reality of this phenomenon. By simulating the schematic of Figure 5 (a) and using an *x-y* grapher with separate *x*-axis (real signal) and *y*-axis (imaginary signal) inputs, we can actually see the dot tracing a circle in the counter-clockwise direction.

The *x-y* graph block plots data in the first input (the *x* direction) against data in the second input (the *y* direction). The clockwise rotation of the dot in the *x-y* grapher gives the notion of negative frequency. It can be verified by making the *y*-axis input (imaginary signal) negative. In this case we trace a circle in the opposite direction which is clockwise direction [Figure 6 (d)].

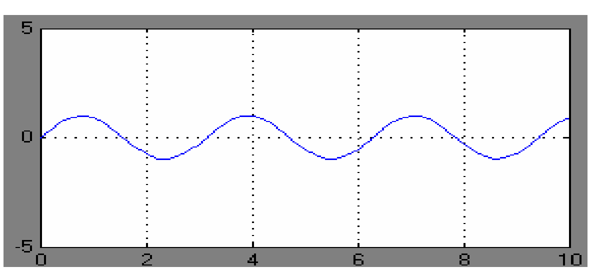
Negative frequencies follow the important rule of symmetry: Negative frequency components are always mirror-images of the positive frequency components. If we are to generate real signals from complex exponentials, complex conjugate pairs of eigen functions are required [8]. So, negative frequencies whether analog or digital are just as "real" as positive frequencies.



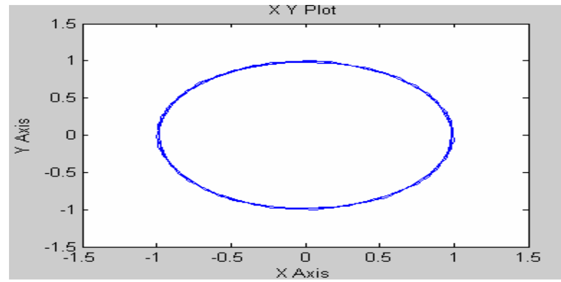
(a)



(b)

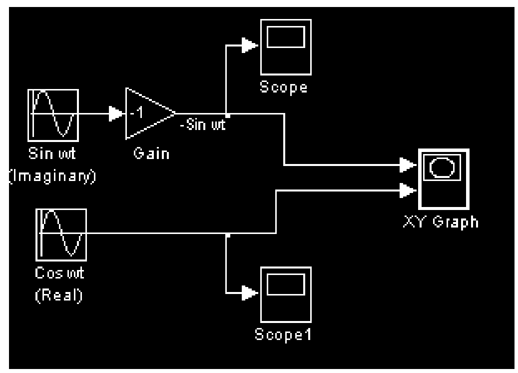


(c)

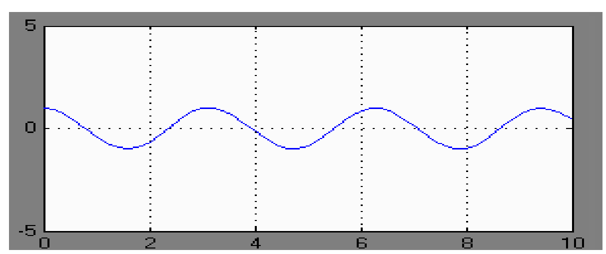


(d)

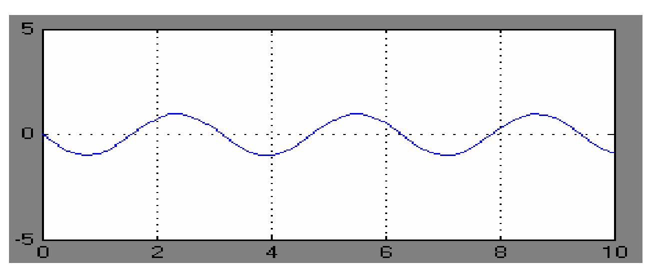
Fig. 5. A schematic showing the existence of positive frequency: (a) Schematic (b) Scope revealing x-axis input (real part of complex signal) which is cos ωt (c) Scope revealing y-axis input (imaginary part of complex) which is sin ωt (d) x-y graph showing a unit circle due to the counter-clockwise rotation of trace.



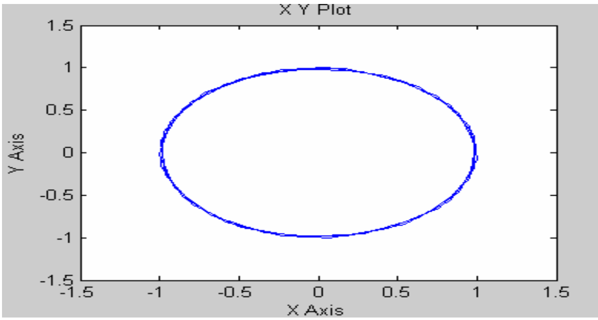
(a)



(b)



(c)



(d)

Fig. 6. A schematic showing the existence of negative frequency: (a) Schematic (b) Scope revealing x-axis input (real part of complex signal) which is cos ωt (c) Scope revealing y-axis input (imaginary part of complex signal) which is -sin ωt (d) x-y graph showing a unit circle due to the clockwise rotation of trace.

4. CONCLUSION

The negative frequency concept can have many flexible details depending on its various aspects. The projections of both the clockwise and anticlockwise phasors of our complex analysis on the real axis are simply added up in time domain giving us the cosine function. So, it is just the shifting from one reference system to other or transformation from one domain to other that highlights its significance. The time domain version of this concept is confusing due to our conventional single dimensional consideration of frequency concept. More insight can be achieved in this argument by treating frequency as a two-dimensional reality. Nevertheless the negative frequency notion is a mathematical necessity and is an outcome of phasor analysis signifying an admirable model for simplifying mathematical and subsequent frequency domain issues.

REFERENCES

[1] C. Langton, "Fourier Analysis Made Easy, Part 2", Signal Processing and Communications Newsletter on website http//www.complextoreal.com

[2] K.W. Martin, "Complex Signal Processing is not Complex", IEEE Trans. Circuits Syst., Vol. 51, No. 9, September 2004.

[3] A.V. Oppenheim and A.L. Willsky, "Signals & Systems", Second Edition, Prentice Hall, 1997.

[4] B. Mulgrew, P. Grant and J.Thompson, "Digital Signal Processing: Concepts and Applications", Second Edition, Palgrave Macmillan, 2003.

[5] Jose Antonio de la O, "New Theory for Phasor Measurement", in IEEE Instrument and Measurement Conference, Anchorge, USA, May 2002.

[6] http//www.dsprelated.com/comp.dsp

[7] P.O. Brackett and G.R. Lang, "Complex Analogue Filters", European Conference on Circuit Theory and Design, Hague, Netherlands, pp. 412 - 419, August 1981.

[8] M. Elfataoui and G. Mirchandani, "Discrete-Time Analytic Signals with Improved Shiftability", IEEE International Conference on Acoustic, Speech and Signal Processing, 2004, Vol 2, pp. 477-480, May 2004.