Electrical impedance

A graphical representation of the complex impedance plane.

Electrical impedance, or simply **impedance**, describes a measure of opposition to alternating current (AC). Electrical impedance extends the concept of resistance to AC circuits, describing not only the relative amplitudes of the voltage and current, but also the relative phases. When the circuit is driven with direct current (DC) there is no distinction between impedance and resistance; the latter can be thought of as impedance with zero phase angle.

The symbol for impedance is usually Z and it may be represented by writing its magnitude and phase in the form $Z\angle\theta$. However, complex number representation is more powerful for circuit analysis purposes. The term *impedance* was coined by Oliver Heaviside in July 1886. Arthur Kennelly was the first to represent impedance with complex numbers in 1893.

Impedance is defined as the frequency domain ratio of the voltage to the current. In other words, it is voltage–current ratio for a single complex exponential at a particular frequency ω. In general, impedance will be a complex number, but this complex number has the same units as resistance, for which the SI unit is the ohm. For a sinusoidal current or voltage input, the polar form of the complex impedance relates the amplitude and phase of the voltage and current. In particular,

- The magnitude of the complex impedance is the ratio of the voltage amplitude to the current amplitude.
- The phase of the complex impedance is the phase shift by which the current is ahead of the voltage.

The reciprocal of impedance is admittance (i.e., admittance is the current-to-voltage ratio, and it conventionally carries mho or Siemens units).

Complex impedance

Impedance is represented as a [complex](#) quantity $\tilde{Z}$ and the term *complex impedance* may be used interchangeably; the [polar form](#) conveniently captures both magnitude and phase characteristics,

$$\tilde{Z} = Z e^{j\theta}$$

where the magnitude Z represents the ratio of the voltage difference amplitude to the current amplitude, while the argument θ gives the phase difference between voltage and current and j is the [imaginary unit](#). In [Cartesian form](#),

$$\tilde{Z} = R + jX$$

where the [real part](#) of impedance is the resistance R and the [imaginary part](#) is the [reactance](#) X.

Where it is required to add or subtract impedances the cartesian form is more convenient, but when quantities are multiplied or divided the calculation becomes simpler if the polar form is used. A circuit calculation, such as finding the total impedance of two impedances in parallel, may require conversion between forms several times during the calculation. Conversion between the forms follows the normal [conversion rules of complex numbers](#).

Ohm's law
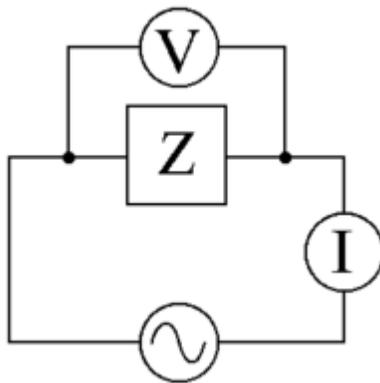
An AC supply applying a voltage V, across a [load](#) Z, driving a current I.

The meaning of electrical impedance can be understood by substituting it into [Ohm's law](#).

$$\tilde{V} = \tilde{I}\tilde{Z} = \tilde{I}Z e^{j\theta}$$

The magnitude of the impedance Z acts just like resistance, giving the drop in voltage amplitude across an impedance $\tilde{Z}$ for a given current $\tilde{I}$. The phase factor tells us that the current lags the voltage by a phase of θ (i.e. in the time domain, the current signal is shifted $\frac{\theta}{2\pi}T$ to the right with respect to the voltage signal).

Just as impedance extends Ohm's law to cover AC circuits, other results from DC circuit analysis such as [voltage division](), [current division](), [Thevenin's theorem](), and [Norton's theorem](), can also be extended to AC circuits by replacing resistance with impedance.

Complex voltage and current
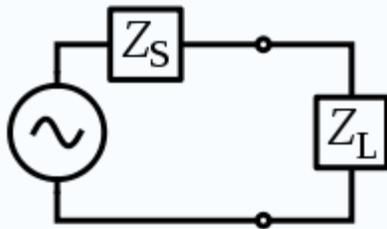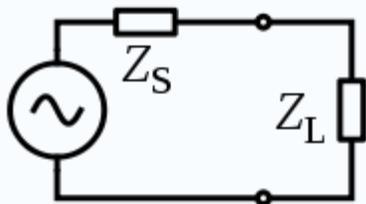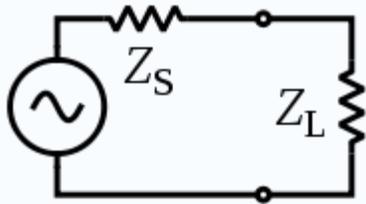
Generalized impedances in a circuit can be drawn with the same symbol as a resistor (US ANSI or DIN Euro) or with a labeled box.

In order to simplify calculations, [sinusoidal]() voltage and current waves are commonly represented as complex-valued functions of time denoted as $\tilde{V}$ and $\tilde{I}$.

$$\tilde{V} = V_0 e^{j(\omega t + \phi_V)}$$
$$\tilde{I} = I_0 e^{j(\omega t + \phi_I)}$$

Impedance is defined as the ratio of these quantities.

$$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}}$$

Substituting these into Ohm's law we have

$$V_0 e^{j(\omega t + \phi_V)} = I_0 e^{j(\omega t + \phi_I)} Z e^{j\theta}$$
$$= I_0 Z e^{j(\omega t + \phi_I + \theta)}$$

Noting that this must hold for all t, we may equate the magnitudes and phases to obtain

$$V_0 = I_0 Z$$
$$\phi_V = \phi_I + \theta$$

The magnitude equation is the familiar Ohm's law applied to the voltage and current amplitudes, while the second equation defines the phase relationship.

Validity of complex representation

This representation using complex exponentials may be justified by noting that (by Euler's formula):

$$\cos(\omega t + \phi) = \frac{1}{2} \left[e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)} \right]$$

i.e. a real-valued sinusoidal function (which may represent our voltage or current waveform) may be broken into two complex-valued functions. By the principle of superposition, we may analyse the behaviour of the sinusoid on the left-hand side by analysing the behaviour of the two complex terms on the right-hand side. Given the symmetry, we only need to perform the analysis for one right-hand term; the results will be identical for the other. At the end of any calculation, we may return to real-valued sinusoids by further noting that

$$\cos(\omega t + \phi) = \Re\left\{ e^{j(\omega t + \phi)} \right\}$$

In other words, we simply take the real part of the result.

Phasors

A phasor is a constant complex number, usually expressed in exponential form, representing the complex amplitude (magnitude and phase) of a sinusoidal function of time. Phasors are used by electrical engineers to simplify computations involving sinusoids, where they can often reduce a differential equation problem to an algebraic one.

The impedance of a circuit element can be defined as the ratio of the phasor voltage across the element to the phasor current through the element, as determined by the relative amplitudes and phases of the voltage and current. This is identical to the definition from Ohm's law given above, recognising that the factors of $e^{j\omega t}$ cancel.

Device examples
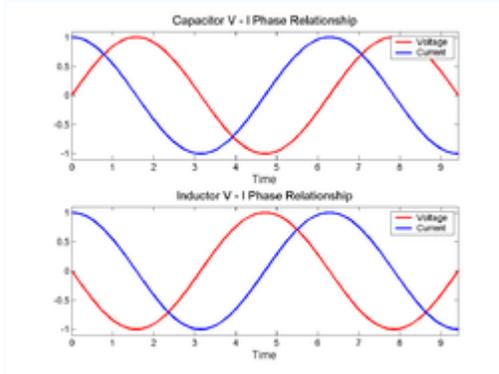
The phase angles in the equations for the impedance of inductors and capacitors indicate that the voltage across a capacitor *lags* the current through it by a phase of $\pi / 2$, while the voltage across an inductor *leads* the current through it by $\pi / 2$. The identical voltage and current amplitudes tell us that the magnitude of the impedance is equal to one.

The impedance of an ideal resistor is purely real and is referred to as a *resistive impedance*:

$$\tilde{Z}_R = R.$$

Ideal inductors and capacitors have a purely imaginary *reactive impedance*:

$$\tilde{Z}_L = j\omega L,$$
$$\tilde{Z}_C = \frac{1}{j\omega C}.$$

Note the following identities for the imaginary unit and its reciprocal:

$$j = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = e^{j\frac{\pi}{2}},$$
$$\frac{1}{j} = -j = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = e^{j(-\frac{\pi}{2})}.$$

Thus we can rewrite the inductor and capacitor impedance equations in polar form:

$$\tilde{Z}_L = \omega L e^{j\frac{\pi}{2}},$$
$$\tilde{Z}_C = \frac{1}{\omega C} e^{j(-\frac{\pi}{2})}.$$

The magnitude tells us the change in voltage amplitude for a given current amplitude through our impedance, while the exponential factors give the phase relationship.

Deriving the device specific impedances

What follows below is a derivation of impedance for each of the three basic circuit elements, the resistor, the capacitor, and the inductor. Although the idea can be extended to define the relationship between the voltage and current of any arbitrary signal, these derivations will assume sinusoidal signals, since any arbitrary signal can be approximated as a sum of sinusoids through Fourier Analysis.

Resistor

For a resistor, we have the relation:

$$v_{\mathrm{R}}\left(t\right) = i_{\mathrm{R}}\left(t\right)R.$$

This is simply a statement of Ohm's Law.

Considering the voltage signal to be

$$v_{\mathrm{R}}(t) = V_p \sin(\omega t),$$

it follows that

$$\frac{v_{\mathrm{R}}\left(t\right)}{i_{\mathrm{R}}\left(t\right)} = \frac{V_p \sin(\omega t)}{I_p \sin\left(\omega t\right)} = R.$$

This tells us that the ratio of AC voltage amplitude to AC current amplitude across a resistor is R, and that the AC voltage leads the AC current across a resistor by 0 degrees.

This result is commonly expressed as

$$Z_{\mathrm{resistor}} = R.$$

Capacitor

For a capacitor, we have the relation:

$$i_{\mathrm{C}}(t) = C\frac{dv_{\mathrm{C}}(t)}{dt}.$$

Considering the voltage signal to be

$$v_{\mathrm{C}}(t) = V_p \sin(\omega t)$$

it follows that

$$\frac{dv_C(t)}{dt} = \omega V_p \cos(\omega t).$$

And thus

$$\frac{v_C(t)}{i_C(t)} = \frac{V_p \sin(\omega t)}{\omega V_p C \cos(\omega t)} = \frac{\sin(\omega t)}{\omega C \sin\left(\omega t + \frac{\pi}{2}\right)}.$$

This tells us that the ratio of AC voltage amplitude to AC current amplitude across a capacitor is $\frac{1}{\omega C}$, and that the AC voltage leads the AC current across a capacitor by -90 degrees (or the AC current leads the AC voltage across a capacitor by 90 degrees).

This result is commonly expressed in polar form, as

$$\tilde{Z}_{capacitor} = \frac{1}{\omega C} e^{-j\frac{\pi}{2}}$$

or, by applying Euler's formula, as

$$\tilde{Z}_{capacitor} = -j\frac{1}{\omega C} = \frac{1}{j\omega C}.$$

Inductor

For the inductor, we have the relation:

$$v_L(t) = L\frac{di_L(t)}{dt}.$$

This time, considering the current signal to be

$$i_L(t) = I_p \sin(\omega t),$$

it follows that

$$\frac{di_L(t)}{dt} = \omega I_p \cos(\omega t).$$

And thus

$$\frac{v_L(t)}{i_L(t)} = \frac{\omega I_p L \cos(\omega t)}{I_p \sin(\omega t)} = \frac{\omega L \sin\left(\omega t + \frac{\pi}{2}\right)}{\sin(\omega t)}.$$

This tells us that the ratio of AC voltage amplitude to AC current amplitude across an inductor is ωL, and that the AC voltage leads the AC current across an inductor by 90 degrees.

This result is commonly expressed in polar form, as

$$\tilde{Z}_{\text{inductor}} = \omega L e^{j\frac{\pi}{2}}.$$

Or, more simply, using Euler's formula, as

$$\tilde{Z}_{\text{inductor}} = j\omega L.$$

Resistance vs reactance

It is important to realize that resistance and reactance are not individually significant; **together** they determine the magnitude and phase of the impedance, through the following relations:

$$|\tilde{Z}| = \sqrt{\tilde{Z}\tilde{Z}^*} = \sqrt{R^2 + X^2}$$
$$\theta = \arctan\left(\frac{X}{R}\right)$$

In many applications the relative phase of the voltage and current is not critical so only the magnitude of the impedance is significant.

Resistance

Resistance R is the real part of impedance; a device with a purely resistive impedance exhibits no phase shift between the voltage and current.

$$R = Z\cos\theta$$

Reactance

Reactance X is the imaginary part of the impedance; a component with a finite reactance induces a phase shift θ between the voltage across it and the current through it.

$$X = Z\sin\theta$$

A purely reactive component is distinguished by the fact that the sinusoidal voltage across the component is in quadrature with the sinusoidal current through the component. This implies that the component alternately absorbs energy from the circuit and then returns energy to the circuit. A pure reactance will not dissipate any power.

Capacitive reactance

A capacitor has a purely reactive impedance which is inversely proportional to the signal frequency. A capacitor consists of two conductors separated by an insulator, also known as a dielectric.

At low frequencies a capacitor is open circuit, as no charge flows in the dielectric. A DC voltage applied across a capacitor causes charge to accumulate on one side; the electric field due to the accumulated charge is the source of the opposition to the current. When the potential associated with the charge exactly balances the applied voltage, the current goes to zero.

Driven by an AC supply, a capacitor will only accumulate a limited amount of charge before the potential difference changes sign and the charge dissipates. The higher the frequency, the less charge will accumulate and the smaller the opposition to the current.

Inductive reactance

An inductor has a purely reactive impedance which is proportional to the signal frequency. An inductor consists of a coiled conductor. Faraday's law of electromagnetic induction gives the back emf $\mathcal{E}$ (voltage opposing current) due to a rate-of-change of magnetic field B through a current loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

For an inductor consisting of a coil with N loops this gives.

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}.$$

The back-emf is the source of the opposition to current flow. A constant direct current has a zero rate-of-change, and sees an inductor as a short-circuit (it is typically made from a material with a low resistivity). An alternating current has a time rate-of-change that is proportional to frequency and so the inductive reactance is proportional to frequency.

Combining impedances

The total impedance of many simple networks of components can be calculated using the rules for combining impedances in series and parallel. The rules are identical to those used for combining resistances, except that the numbers in general will be complex numbers. In the general case however, equivalent impedance transforms in addition to series and parallel will be required.

Series combination

For components connected in series, the current through each circuit element is the same; the ratio of voltages across any two elements is the inverse ratio of their impedances.

$$\tilde{Z}_{eq} = \tilde{Z}_1 + \tilde{Z}_2 + ... + \tilde{Z}_n = (R_1 + R_2 + ... + R_n) + j(X_1 + X_2 + ... + X_n)$$

Parallel combination

For components connected in parallel, the voltage across each circuit element is the same; the ratio of currents through any two elements is the inverse ratio of their impedances.
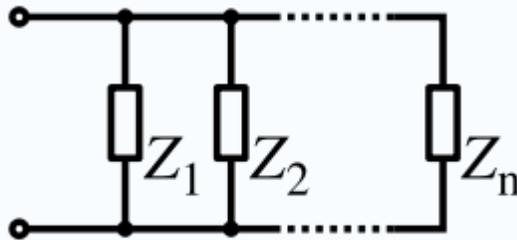
$$\frac{1}{\tilde{Z}_{eq}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + .. + \frac{1}{\tilde{Z}_n}$$

or, when n = 2:

$$\tilde{Z}_{eq} = \tilde{Z}_1 \| \tilde{Z}_2 \stackrel{\text{def}}{=} \frac{\tilde{Z}_1 \tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}$$

The equivalent impedance $\tilde{Z}_{eq}$ can be calculated in terms of the equivalent resistance R_{eq} and reactance X_{eq}.

$$\tilde{Z}_{eq} = R_{eq} + jX_{eq}$$
$$R_{eq} = \frac{(X_1 R_2 + X_2 R_1)(X_1 + X_2) + (R_1 R_2 - X_1 X_2)(R_1 + R_2)}{(R_1 + R_2)^2 + (X_1 + X_2)^2}$$

$$X_{eq} = \frac{(X_1 R_2 + X_2 R_1)(R_1 + R_2) - (R_1 R_2 - X_1 X_2)(X_1 + X_2)}{(R_1 + R_2)^2 + (X_1 + X_2)^2}$$

Measuring Impedance

According to Ohm's law the impedance of a device can be calculated by complex division of the voltage and current. The impedance of the device can be calculated by applying a sinusoidal voltage to the device in series with a resistor, and measuring the voltage across the resistor and across the device. Performing this measurement by sweeping the frequencies of the applied signal provides the impedance phase and magnitude.