Lesson 8

Integral Properties and Average Value

Properties of Integrals

- Can be viewed as area properties
- Or as properties of Riemann sums

Additive Property

$$\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx$$





Sums and Differences

$$\int_{a}^{b} \left[f(x) \pm g(x) \right] dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx$$

The integral is a sum, and we can always rearrange sums.

Constant Multiples
$$\int_{a}^{b} c f(x) dx = c \int_{a}^{b} f(x) dx$$

Can also factor out a constant multiple from a sum.

Two Conventions

$$1. \quad \int_{a}^{a} f(x) \, dx = 0$$



2. If
$$a < b$$
,
$$\int_{b}^{a} f(x) dx = -\int_{a}^{b} f(x) dx$$



Four Examples

Given:

$$\int_0^2 f(x) dx = 4$$
$$\int_0^1 f(x) dx = 1$$
$$\int_1^2 g(x) dx = -1$$
$$\int_0^1 g(x) dx = 3$$

This we *cannot do*; there is no rule for products!

Evaluate, if possible:

$$\int_{0}^{1} 2f(x) + g(x) dx = 5$$
$$\int_{0}^{2} g(x) dx = 2$$
$$\int_{1}^{2} -5f(x) dx = -15$$

g(x) dx =

Four Examples

Given:

Note:

$$\int_0^2 f(x) dx = 4$$
$$\int_0^1 f(x) dx = 1$$
$$\int_1^2 g(x) dx = -1$$
$$\int_0^1 g(x) dx = 3$$

$$\int_{a}^{b} f(x) dx + \int_{b}^{c} f(x) dx = \int_{a}^{c} f(x) dx$$
so
$$\int_{a}^{c} f(x) dx - \int_{b}^{c} f(x) dx = \int_{a}^{b} f(x) dx$$

Average Value

We will now give a definition for the average value of a function on an interval.

•You already know about functions and how to take the average of some finite set. i.e. Average of a_1, a_2, \dots, a_n is

$$=\frac{a_1+a_2+\cdots+a_n}{n}$$

•Before I show how to do this...let's talk about WHY we might want to do this?

Consider the following picture:



• How high would the water level be if the waves all settled?

Okay! So, now that you have seen that this an interesting question...

Let's forget about real life, and... Do Some Math Suppose we have a "nice" function and we need to find its average value over the interval [a,b].



Average Value of a Function

Try this, to *approximate* the average:

- Split [*a*, *b*] into subintervals width $\Delta x = (b a)/n$.
- Pick a point x_i in each subinterval.
- Average just the $f(x_i)$:

$$=\frac{f(x_{1}) + f(x_{2}) + \dots + f(x_{n})}{n}$$

estimate for the average value

As *n* increases, the approximation gets better.

Sounds and looks a bit like an integral!

Average Value of a Function

Now, get Δx into the mix:



In a more condensed form, we now get:

average
$$\approx \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x$$

But we want to get out of the finite, and into the infinite!

How do we do this?

Take Limits!!!

In this way, we get the average value of f(x) over the interval [a,b]: $average = \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i) \Delta x = \frac{1}{b-a} \int_{a}^{b} f(x) dx$

So, if f is a "nice" function (i.e. we can compute its integral) then we have a precise solution to our problem.

Average Value of a Function

So we make the following definition: The average value of a function f(x) on an interval [a, b] is

$$\frac{1}{b-a}\int_{a}^{b}f(x)\,dx$$

Let's look back at our graph:



Now we can answer our fish tank question!

(That is, if the waves were described by an integrable function)

• How high would the water level be if the waves all settled?



Find the average value of the function $f(x) = x^2$ on the interval [-1, 1].

By definition, the average value is

$$\frac{1}{b-a}\int_{a}^{b}f(x)\,dx = \frac{1}{1-(-1)}\int_{-1}^{1}x^{2}\,dx = \frac{1}{2}\int_{-1}^{1}x^{2}\,dx$$

Now $\int_{-1}^{1} x^2 dx = 2/3$ (Simpson's Rule)

So the average value is $\frac{1}{2} \int_{-1}^{1} x^2 \, dx = (\frac{1}{2})(\frac{2}{3}) = \frac{1}{3}$

The average value of the function $f(x) = x^2$ on the interval [-1, 1] is 1/3.



Find the average value of $f(t) = 2^t$ on the interval [0, 2]. First, make a graphical estimate:



Find the average value of $f(t) = 2^t$ on the interval [0, 2].

Now, calculate:

$$\frac{1}{2-0}\int_0^2 2^t dt = \frac{1}{2}\int_0^2 2^t dt$$

Use Simpson to approximate the integral:

$$\int_{0}^{2} 2^{t} dt \approx \frac{1/2}{3} \Big[2^{0} + (4)(2^{1/2}) + (2)(2^{1}) + (4)(2^{3/2}) + 2^{2} \Big]$$
$$\approx 4.328$$

So the average value is about (1/2)(4.328) or 2.164:

Find the average value of $f(t) = 2^t$ on the interval [0, 2]. So the average value is about (1/2)(4.328) or 2.164:



Summary

- Discussed algebraic properties of integrals, and used these to calculate integrals.
- Motivated (and stated) the formula for the average value of *f* on [*a*, *b*]:

$$\frac{1}{b-a}\int_{a}^{b}f(x)\,dx$$

- Calculated average values.
- Estimated the average value from a graph.
- Interpreted the average value graphically.

In Class Group work

1. What is the average value of f(x) = 7 on the interval [2, 6]?

2. Suppose the following integrals are known :

$$\int_{a}^{b} f(x) dx = 4 \qquad \int_{a}^{b} g(x) dx = -2$$
$$\int_{a}^{b} (f(x))^{2} dx = 5 \qquad \int_{a}^{b} (g(x))^{2} dx = 7$$

Find the integrals

•
$$\int_{a}^{a} f(x)g(x)dx$$

• $\int_{b}^{a} -f(x) + 3g(x)dx$
• $\int_{b}^{b} (f(x))^{2} dx - 2\left(\int_{a}^{b} g(x)dx\right)^{2}$