Chapter 3

FORWARD KINEMATICS: THE DENAVIT-HARTENBERG CONVENTION

In this chapter we develop the **forward** or **configuration kinematic equations** for rigid robots. The forward kinematics problem is concerned with the relationship between the individual joints of the robot manipulator and the position and orientation of the tool or end-effector. Stated more formally, the forward kinematics problem is to determine the position and orientation of the end-effector, given the values for the joint variables of the robot. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension in the case of prismatic or sliding joints. The forward kinematics problem is to be contrasted with the **inverse** kinematics problem, which will be studied in the next chapter, and which is concerned with determining values for the joint variables that achieve a desired position and orientation for the end-effector of the robot.

3.1 Kinematic Chains

As described in Chapter 1, a robot manipulator is composed of a set of links connected together by various joints. The joints can either be very simple, such as a revolute joint or a prismatic joint, or else they can be more complex, such as a ball and socket joint. (Recall that a revolute joint is like a hinge and allows a relative rotation about a single axis, and a prismatic joint permits a linear motion along a single axis, namely an extension or retraction.) The difference between the two situations is that, in the first instance, the joint has only a single degree-of-freedom of motion: the angle of rotation in the case of a revolute joint, and the amount of linear displacement in the case of a prismatic joint. In contrast, a ball and socket joint has two degrees-of-freedom. In this book it is assumed throughout that all joints have only a single degree-of-freedom. Note that the assumption

does not involve any real loss of generality, since joints such as a ball and socket joint (two degrees-of-freedom) or a spherical wrist (three degrees-of-freedom) can always be thought of as a succession of single degree-of-freedom joints with links of length zero in between.

With the assumption that each joint has a single degree-of-freedom, the action of each joint can be described by a single real number: the angle of rotation in the case of a revolute joint or the displacement in the case of a prismatic joint. The objective of forward kinematic analysis is to determine the *cumulative* effect of the entire set of joint variables. In this chapter we will develop a set of conventions that provide a systematic procedure for performing this analysis. It is, of course, possible to carry out forward kinematics analysis even without respecting these conventions, as we did for the two-link planar manipulator example in Chapter 1. However, the kinematic analysis of an n-link manipulator can be extremely complex and the conventions introduced below simplify the analysis considerably. Moreover, they give rise to a universal language with which robot engineers can communicate.

A robot manipulator with n joints will have $n+1$ links, since each joint connects two links. We number the joints from 1 to n, and we number the links from 0 to n, starting from the base. By this convention, joint i connects link $i-1$ to link i. We will consider the location of joint i to be fixed with respect to link $i-1$. *When joint i is actuated, link i moves.* Therefore, link 0 (the first link) is fixed, and does not move when the joints are actuated. Of course the robot manipulator could itself be mobile (e.g., it could be mounted on a mobile platform or on an autonomous vehicle), but we will not consider this case in the present chapter, since it can be handled easily by slightly extending the techniques presented here.

With the i^{th} joint, we associate a *joint variable*, denoted by q_i. In the case of a revolute joint, q_i is the angle of rotation, and in the case of a prismatic joint, q_i is the joint displacement:

$$q_i = \begin{cases} \theta_i & : \text{ joint i revolute} \\ d_i & : \text{ joint i prismatic} \end{cases} . \qquad (3.1)$$

To perform the kinematic analysis, we rigidly attach a coordinate frame to each link. In particular, we attach $o_i x_i y_i z_i$ to link i. This means that, whatever motion the robot executes, the coordinates of each point on link i are constant when expressed in the i^{th} coordinate frame. Furthermore, when joint i is actuated, link i and its attached frame, $o_i x_i y_i z_i$, experience a resulting motion. The frame $o_0 x_0 y_0 z_0$, which is attached to the robot base, is referred to as the inertial frame. Figure 3.1 illustrates the idea of attaching frames rigidly to links in the case of an elbow manipulator.

Now suppose A_i is the homogeneous transformation matrix that expresses the position and orientation of $o_i x_i y_i z_i$ with respect to $o_{i-1} x_{i-1} y_{i-1} z_{i-1}$. The matrix A_i is not constant, but varies as the configuration of the robot is changed. However, the assumption that all joints are either revolute or prismatic means that A_i is a function of only a single joint variable, namely q_i. In other words,

$$A_i = A_i(q_i). \qquad (3.2)$$

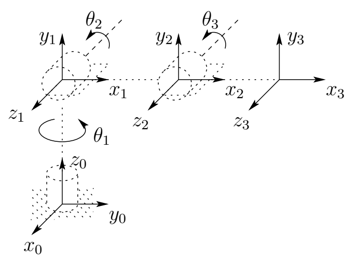

Figure 3.1: Coordinate frames attached to elbow manipulator.

Now the homogeneous transformation matrix that expresses the position and orientation of $o_jx_jy_jz_j$ with respect to $o_ix_iy_iz_i$ is called, by convention, a **transformation matrix**, and is denoted by T^i_j. From Chapter 2 we see that

$$\begin{aligned} T^i_j &= A_{i+1}A_{i+2}\dots A_{j-1}A_j \text{ if } i < j \\ T^i_j &= I \text{ if } i = j \\ T^i_j &= (T^j_i)^{-1} \text{ if } j > i. \end{aligned} \tag{3.3}$$

By the manner in which we have rigidly attached the various frames to the corresponding links, it follows that the position of any point on the end-effector, when expressed in frame n, is a constant independent of the configuration of the robot. Denote the position and orientation of the end-effector with respect to the inertial or base frame by a three-vector o^0_n (which gives the coordinates of the origin of the end-effector frame with respect to the base frame) and the 3×3 rotation matrix R^0_n, and define the homogeneous transformation matrix

$$H = \begin{bmatrix} R^0_n & o^0_n \\ 0 & 1 \end{bmatrix}. \tag{3.4}$$

Then the position and orientation of the end-effector in the inertial frame are given by

$$H = T^0_n = A_1(q_1)\cdots A_n(q_n). \tag{3.5}$$

Each homogeneous transformation A_i is of the form

$$A_i = \begin{bmatrix} R^{i-1}_i & o^{i-1}_i \\ 0 & 1 \end{bmatrix}. \tag{3.6}$$

Hence

$$T^i_j = A_{i+1}\cdots A_j = \begin{bmatrix} R^i_j & o^i_j \\ 0 & 1 \end{bmatrix}. \tag{3.7}$$

The matrix R^i_j expresses the orientation of $o_jx_jy_jz_j$ relative to $o_ix_iy_iz_i$ and is given by the rotational parts of the A-matrices as

$$R^i_j = R^i_{i+1}\cdots R^{j-1}_j. \tag{3.8}$$

The coordinate vectors o^i_j are given recursively by the formula

$$o^i_j = o^i_{j-1} + R^i_{j-1}o^{j-1}_j, \tag{3.9}$$

These expressions will be useful in Chapter 5 when we study Jacobian matrices.

In principle, that is all there is to forward kinematics! Determine the functions $A_i(q_i)$, and multiply them together as needed. However, it is possible to achieve a considerable amount of streamlining and simplification by introducing further conventions, such as the Denavit-Hartenberg representation of a joint, and this is the objective of the remainder of the chapter.

3.2 Denavit Hartenberg Representation

While it is possible to carry out all of the analysis in this chapter using an arbitrary frame attached to each link, it is helpful to be systematic in the choice of these frames. A commonly used convention for selecting frames of reference in robotic applications is the Denavit-Hartenberg, or D-H convention. In this convention, each homogeneous transformation A_i is represented as a product of four basic transformations

$$\begin{aligned} A_i &= Rot_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}Rot_{x,\alpha_i} \qquad (3.10) \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_ic_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_is_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where the four quantities θ_i, a_i, d_i, α_i are parameters associated with link i and joint i. The four parameters a_i, α_i, d_i, and θ_i in (3.10) are generally given the names **link length**, **link twist**, **link offset**, and **joint angle**, respectively. These names derive from specific aspects of the geometric relationship between two coordinate frames, as will become apparent below. Since the matrix A_i is a function of a single variable, it turns out that three of the above four quantities are constant for a given link, while the fourth parameter, θ_i for a revolute joint and d_i for a prismatic joint, is the joint variable.

From Chapter 2 one can see that an arbitrary homogeneous transformation matrix can be characterized by six numbers, such as, for example, three numbers to specify the

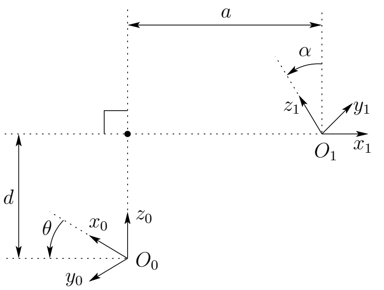

Figure 3.2: Coordinate frames satisfying assumptions DH1 and DH2.

fourth column of the matrix and three Euler angles to specify the upper left 3×3 rotation matrix. In the D-H representation, in contrast, there are only *four* parameters. How is this possible? The answer is that, while frame i is required to be rigidly attached to link i, we have considerable freedom in choosing the origin and the coordinate axes of the frame. For example, it is not necessary that the origin, o_i, of frame i be placed at the physical end of link i. In fact, it is not even necessary that frame i be placed within the physical link; frame i could lie in free space — so long as frame i is *rigidly attached* to link i. By a clever choice of the origin and the coordinate axes, it is possible to cut down the number of parameters needed from six to four (or even fewer in some cases). In Section 3.2.1 we will show why, and under what conditions, this can be done, and in Section 3.2.2 we will show exactly how to make the coordinate frame assignments.

3.2.1 Existence and uniqueness issues

Clearly it is not possible to represent any arbitrary homogeneous transformation using only four parameters. Therefore, we begin by determining just which homogeneous transformations can be expressed in the form (3.10). Suppose we are given two frames, denoted by frames 0 and 1, respectively. Then there exists a unique homogeneous transformation matrix A that takes the coordinates from frame 1 into those of frame 0. Now suppose the two frames have two additional features, namely:

(DH1) The axis x_1 is perpendicular to the axis z_0

(DH2) The axis x_1 intersects the axis z_0

as shown in Figure 3.2. Under these conditions, we claim that there exist unique numbers a, d, θ, α such that

$$A = Rot_{z,\theta}\text{Trans}_{z,d}\text{Trans}_{x,a}Rot_{x,\alpha}. \tag{3.11}$$

Of course, since θ and α are angles, we really mean that they are unique to within a multiple of 2π. To show that the matrix A can be written in this form, write A as

$$A = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{bmatrix} \tag{3.12}$$

and let $\boldsymbol{r}_i$ denote the i^{th} column of the rotation matrix R_1^0. We will now examine the implications of the two DH constraints.

If (DH1) is satisfied, then x_1 is perpendicular to z_0 and we have $x_1 \cdot z_0 = 0$. Expressing this constraint with respect to $o_0x_0y_0z_0$, using the fact that $\boldsymbol{r}_1$ is the representation of the unit vector x_1 with respect to frame 0, we obtain

$$\begin{aligned} 0 &= x_1^0 \cdot z_0^0 & (3.13)\\ &= [r_{11}, r_{21}, r_{31}]^T \cdot [0,0,1]^T & (3.14)\\ &= r_{31}. & (3.15)\end{aligned}$$

Since $r_{31} = 0$, we now need only show that there exist *unique* angles θ and α such that

$$R_1^0 = R_{x,\theta}R_{x,\alpha} = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}. \tag{3.16}$$

The only information we have is that $r_{31} = 0$, but this is enough. First, since each row and column of R_1^0 must have unit length, $r_{31} = 0$ implies that

$$\begin{aligned} r_{11}^2 + r_{21}^2 &= 1, \\ r_{32}^2 + r_{33}^2 &= 1 \end{aligned} \tag{3.17}$$

Hence there exist unique θ, α such that

$$(r_{11}, r_{21}) = (c_\theta, s_\theta), \qquad (r_{33}, r_{32}) = (c_\alpha, s_\alpha). \tag{3.18}$$

Once θ and α are found, it is routine to show that the remaining elements of R_1^0 must have the form shown in (3.16), using the fact that R_1^0 is a rotation matrix.

Next, assumption (DH2) means that the displacement between o_0 and o_1 can be expressed as a linear combination of the vectors z_0 and x_1. This can be written as $o_1 = o_0 + dz_0 + ax_1$. Again, we can express this relationship in the coordinates of $o_0x_0y_0z_0$, and we obtain

$$\begin{aligned} o_1^0 &= o_0^0 + dz_0^0 + ax_1^0 & (3.19)\\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a\begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix} & (3.20)\\ &= \begin{bmatrix} ac_\theta \\ as_\theta \\ d \end{bmatrix}. & (3.21)\end{aligned}$$

Combining the above results, we obtain (3.10) as claimed. Thus, we see that four parameters are sufficient to specify any homogeneous transformation that satisfies the constraints (DH1) and (DH2).

Now that we have established that each homogeneous transformation matrix satisfying conditions (DH1) and (DH2) above can be represented in the form (3.10), we can in fact give a physical interpretation to each of the four quantities in (3.10). The parameter a is the distance between the axes z_0 and z_1, and is measured along the axis x_1. The angle α is the angle between the axes z_0 and z_1, measured in a plane normal to x_1. The positive sense for α is determined from z_0 to z_1 by the right-hand rule as shown in Figure 3.3. The

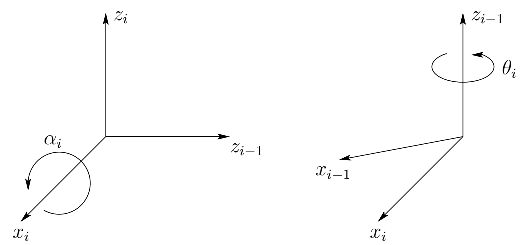

Figure 3.3: Positive sense for α_i and θ_i.

parameter d is the distance between the origin o_0 and the intersection of the x_1 axis with z_0 measured along the z_0 axis. Finally, θ is the angle between x_0 and x_1 measured in a plane normal to z_0. These physical interpretations will prove useful in developing a procedure for assigning coordinate frames that satisfy the constraints (DH1) and (DH2), and we now turn our attention to developing such a procedure.

3.2.2 Assigning the coordinate frames

For a given robot manipulator, one can always choose the frames $0, \ldots, n$ in such a way that the above two conditions are satisfied. In certain circumstances, this will require placing the origin o_i of frame i in a location that may not be intuitively satisfying, but typically this will not be the case. In reading the material below, it is important to keep in mind that the choices of the various coordinate frames are not unique, even when constrained by the requirements above. Thus, it is possible that different engineers will derive differing, but equally correct, coordinate frame assignments for the links of the robot. It is very important to note, however, that the end result (i.e., the matrix T_n^0) will be the same, regardless of the assignment of intermediate link frames (assuming that the coordinate frames for link n coincide). We will begin by deriving the general procedure. We will then discuss various common special cases where it is possible to further simplify the homogeneous transformation matrix.

To start, note that the choice of z_i is arbitrary. In particular, from (3.16), we see that by choosing α_i and θ_i appropriately, we can obtain any arbitrary direction for z_i. Thus, for our first step, we assign the axes $z_0, \ldots, z_{n-1}$ in an intuitively pleasing fashion. Specifically, we assign z_i to be the axis of actuation for joint $i+1$. Thus, z_0 is the axis of actuation for joint 1, z_1 is the axis of actuation for joint 2, etc. There are two cases to consider: (i) if joint $i+1$ is revolute, z_i is the axis of revolution of joint $i+1$; (ii) if joint $i+1$ is prismatic, z_i is the axis of translation of joint $i+1$. At first it may seem a bit confusing to associate z_i with joint $i+1$, but recall that this satisfies the convention that we established in Section 3.1, namely that joint i is fixed with respect to frame i, and that when joint i is actuated, link i and its attached frame, $o_i x_i y_i z_i$, experience a resulting motion.

Once we have established the z-axes for the links, we establish the base frame. The choice of a base frame is nearly arbitrary. We may choose the origin o_0 of the base frame to be any point on z_0. We then choose x_0, y_0 in any convenient manner so long as the resulting frame is right-handed. This sets up frame 0.

Once frame 0 has been established, we begin an iterative process in which we define frame i using frame $i-1$, beginning with frame 1. Figure 3.4 will be useful for understanding the process that we now describe.

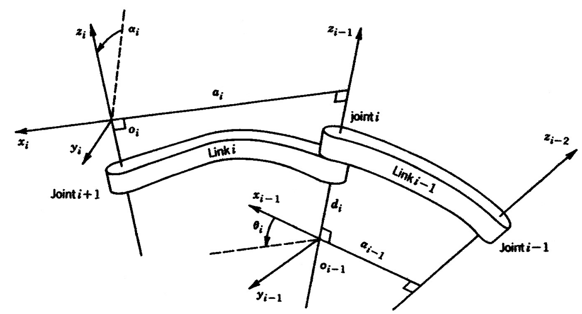

Figure 3.4: Denavit-Hartenberg frame assignment.

In order to set up frame i it is necessary to consider three cases: (i) the axes z_{i-1}, z_i are not coplanar, (ii) the axes z_{i-1}, z_i intersect (iii) the axes z_{i-1}, z_i are parallel. Note that in both cases (ii) and (iii) the axes z_{i-1} and z_i are coplanar. This situation is in fact quite common, as we will see in Section 3.3. We now consider each of these three cases.

(i) z_{i-1} and z_i are not coplanar: If z_{i-l} and z_i are not coplanar, then there exists a *unique* line segment perpendicular to both z_{i-1} and z_i such that it connects both lines and it has minimum length. The line containing this common normal to z_{i-1} and z_i defines x_i, and the point where this line intersects z_i is the origin o_i. By construction, both conditions (DH1) and (DH2) are satisfied and the vector from o_{i-1} to o_i is a linear combination of z_{i-1} and x_i. The specification of frame i is completed by choosing the axis y_i to form a right-hand frame. Since assumptions (DH1) and (DH2) are satisfied the homogeneous transformation matrix A_i is of the form (3.10).

(ii) z_{i-1} is parallel to z_i: If the axes z_{i-1} and z_i are parallel, then there are infinitely many common normals between them and condition (DH1) does not specify x_i completely. In this case we are free to choose the origin o_i anywhere along z_i. One often chooses o_i to simplify the resulting equations. The axis x_i is then chosen either to be directed from o_i toward z_{i-1}, along the common normal, or as the opposite of this vector. A common method for choosing o_i is to choose the normal that passes through o_{i-1} as the x_i axis; o_i is then the point at which this normal intersects z_i. In this case, d_i would be equal to zero. Once x_i is fixed, y_i is determined, as usual by the right hand rule. Since the axes z_{i-1} and z_i are parallel, α_i will be zero in this case.

(iii) z_{i-1} intersects z_i: In this case x_i is chosen normal to the plane formed by z_i and z_{i-1}. The positive direction of x_i is arbitrary. The most natural choice for the origin o_i in this case is at the point of intersection of z_i and z_{i-1}. However, any convenient point along the axis z_i suffices. Note that in this case the parameter a_i equals 0.

This constructive procedure works for frames $0, \ldots, n-l$ in an n-link robot. To complete the construction, it is necessary to specify frame n. The final coordinate system $o_n x_n y_n z_n$ is commonly referred to as the **end-effector** or **tool frame** (see Figure 3.5). The origin

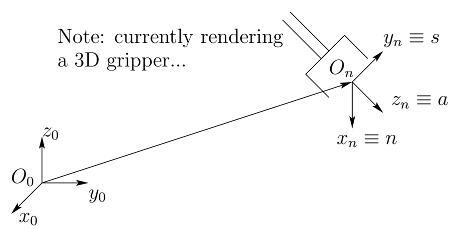

Figure 3.5: Tool frame assignment.

o_n is most often placed symmetrically between the fingers of the gripper. The unit vectors along the x_n, y_n, and z_n axes are labeled as $\boldsymbol{n}$, $\boldsymbol{s}$, and $\boldsymbol{a}$, respectively. The terminology arises from fact that the direction $\boldsymbol{a}$ is the **approach** direction, in the sense that the gripper typically approaches an object along the $\boldsymbol{a}$ direction. Similarly the $\boldsymbol{s}$ direction is the **sliding** direction, the direction along which the fingers of the gripper slide to open and close, and $\boldsymbol{n}$ is the direction **normal** to the plane formed by $\boldsymbol{a}$ and $\boldsymbol{s}$.

In contemporary robots the final joint motion is a rotation of the end-effector by θ_n and the final two joint axes, z_{n-1} and z_n, coincide. In this case, the transformation between the final two coordinate frames is a translation along z_{n-1} by a distance d_n followed (or preceded) by a rotation of θ_n radians about z_{n-1}. This is an important observation that will simplify the computation of the inverse kinematics in the next chapter.

Finally, note the following important fact. In all cases, whether the joint in question is revolute or prismatic, the quantities a_i and α_i are always constant for all i and are characteristic of the manipulator. If joint i is prismatic, then θ_i is also a constant, while d_i is the i^{th} joint variable. Similarly, if joint i is revolute, then d_i is constant and θ_i is the i^{th} joint variable.

3.2.3 Summary

We may summarize the above procedure based on the D-H convention in the following algorithm for deriving the forward kinematics for any manipulator.

Step l: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin o_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i (see Figure 3.3).

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 3.3). θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

3.3 Examples

In the D-H convention the only variable angle is θ, so we simplify notation by writing c_i for $\cos\theta_i$, etc. We also denote $\theta_1 + \theta_2$ by θ_{12}, and $\cos(\theta_1 + \theta_2)$ by c_{12}, and so on. In the following examples it is important to remember that the D-H convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

Example 3.1 Planar Elbow Manipulator

Consider the two-link planar arm of Figure 3.6. The joint axes z_0 and z_1 are normal to

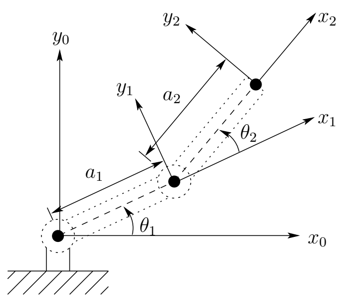

Figure 3.6: Two-link planar manipulator. The z-axes all point out of the page, and are not shown in the figure.

the page. We establish the base frame $o_0x_0y_0z_0$ as shown. The origin is chosen at the point of intersection of the z_0 axis with the page and the direction of the x_0 axis is completely arbitrary. Once the base frame is established, the $o_1x_1y_1z_1$ frame is fixed as shown by the D-H convention, where the origin o_1 has been located at the intersection of z_1 and the page. The final frame $o_2x_2y_2z_2$ is fixed by choosing the origin o_2 at the end of link 2 as shown. The link parameters are shown in Table 3.1. The A-matrices are determined from (3.10) as

Table 3.1: Link parameters for 2-link planar manipulator.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.22}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.23}$$

The T-matrices are thus given by

$$T_1^0 = A_1. \tag{3.24}$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.25}$$

Notice that the first two entries of the last column of T_2^0 are the x and y components of the origin o_2 in the base frame; that is,

$$\begin{aligned} x &= a_1c_1 + a_2c_{12} \\ y &= a_1s_1 + a_2s_{12} \end{aligned} \tag{3.26}$$

are the coordinates of the end-effector in the base frame. The rotational part of T_2^0 gives the orientation of the frame $o_2x_2y_2z_2$ relative to the base frame.

◇

Example 3.2 Three-Link Cylindrical Robot

Consider now the three-link cylindrical robot represented symbolically by Figure 3.7. We establish o_0 as shown at joint 1. Note that the placement of the origin o_0 along z_0 as well as the direction of the x_0 axis are arbitrary. Our choice of o_0 is the most natural, but o_0 could just as well be placed at joint 2. The axis x_0 is chosen normal to the page. Next, since z_0 and z_1 coincide, the origin o_1 is chosen at joint 1 as shown. The x_1 axis is normal to the page when $\theta_1 = 0$ but, of course its direction will change since θ_1 is variable. Since z_2 and z_1 intersect, the origin o_2 is placed at this intersection. The direction of x_2 is chosen

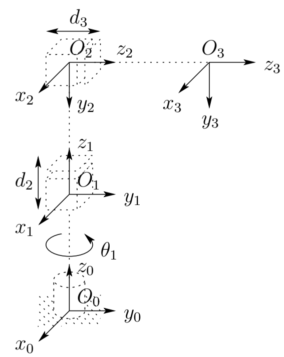

Figure 3.7: Three-link cylindrical manipulator.

Table 3.2: Link parameters for 3-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

parallel to x_1 so that θ_2 is zero. Finally, the third frame is chosen at the end of link 3 as shown.

The link parameters are now shown in Table 3.2. The corresponding A and T matrices

are

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.27)$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1A_2A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1d_3 \\ s_1 & 0 & c_1 & c_1d_3 \\ 0 & -1 & 0 & d_1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.28)$$

◇

Example 3.3 Spherical Wrist

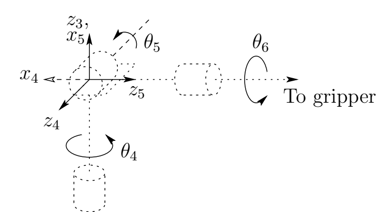

Figure 3.8: The spherical wrist frame assignment.

The spherical wrist configuration is shown in Figure 3.8, in which the joint axes z_3, z_4, z_5 intersect at o. The Denavit-Hartenberg parameters are shown in Table 3.3. The Stanford manipulator is an example of a manipulator that possesses a wrist of this type. In fact, the following analysis applies to virtually all spherical wrists.

We show now that the final three joint variables, θ_4, θ_5, θ_6 are the Euler angles ϕ, θ, ψ, respectively, with respect to the coordinate frame $o_3x_3y_3z_3$. To see this we need only compute

Table 3.3: DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

the matrices A_4, A_5, and A_6 using Table 3.3 and the expression (3.10). This gives

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.29}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.30}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.31}$$

Multiplying these together yields

$$\begin{aligned} T_6^3 = A_4A_5A_6 &= \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.32}$$

Comparing the rotational part R_6^3 of T_6^3 with the Euler angle transformation (2.51) shows that $\theta_4, \theta_5, \theta_6$ can indeed be identified as the Euler angles ϕ, θ and ψ with respect to the coordinate frame $o_3x_3y_3z_3$.

◇

Example 3.4 Cylindrical Manipulator with Spherical Wrist

Suppose that we now attach a spherical wrist to the cylindrical manipulator of Example 3.3.2 as shown in Figure 3.9. Note that the axis of rotation of joint 4 is parallel to z_2 and thus coincides with the axis z_3 of Example 3.3.2. The implication of this is that we can

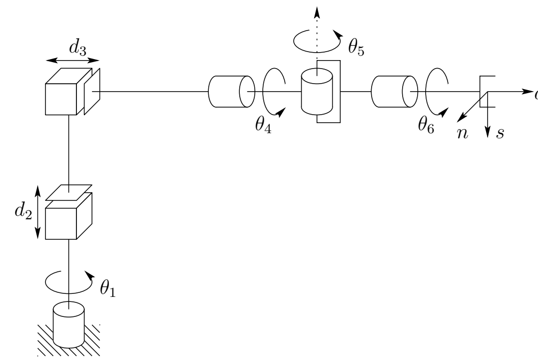

Figure 3.9: Cylindrical robot with spherical wrist.

immediately combine the two previous expression (3.28) and (3.32) to derive the forward kinematics as

$$T_6^0 = T_3^0 T_6^3 \tag{3.33}$$

with T_3^0 given by (3.28) and T_6^3 given by (3.32). Therefore the forward kinematics of this manipulator is described by

$$\begin{aligned} T_6^0 &= \begin{bmatrix} c_1 & 0 & -s_1 & -s_1d_1 \\ s_1 & 0 & c_1 & c_1d_3 \\ 0 & -1 & 0 & d_1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2.
\end{aligned}
$$

Notice how most of the complexity of the forward kinematics for this manipulator results from the orientation of the end-effector while the expression for the arm position from (3.28) is fairly simple. The spherical wrist assumption not only simplifies the derivation of the forward kinematics here, but will also greatly simplify the inverse kinematics problem in the next chapter.

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Example 3.5 Stanford Manipulator

Consider now the Stanford Manipulator shown in Figure 3.10. This manipulator is an

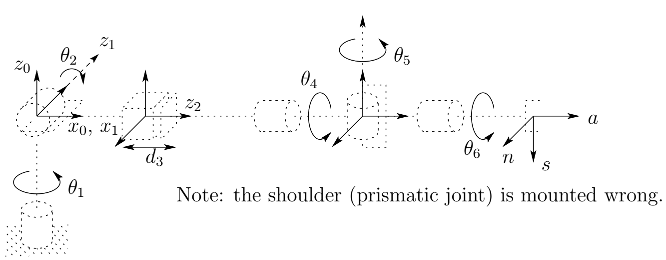

Figure 3.10: DH coordinate frame assignment for the Stanford manipulator.

example of a spherical (RRP) manipulator with a spherical wrist. This manipulator has an offset in the shoulder joint that slightly complicates both the forward and inverse kinematics problems.

Table 3.4: DH parameters for Stanford Manipulator.

Link	d_i	a_i	α_i	θ_i
1	0	0	−90	⋆
2	d_2	0	+90	⋆
3	⋆	0	0	0
4	0	0	−90	⋆
5	0	0	+90	⋆
6	d_6	0	0	⋆

* joint variable

We first establish the joint coordinate frames using the D-H convention as shown. The link parameters are shown in the Table 3.4.

It is straightforward to compute the matrices A_i as

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.35)$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.36)$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.37)$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.38)$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.39)$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.40)$$

T_6^0 is then given as

$$T_6^0 = A_1 \cdots A_6 \tag{3.41}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.42}$$

where

$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \qquad (3.43) \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5). \qquad (3.44)
\end{aligned}$$

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Example 3.6 SCARA Manipulator

As another example of the general procedure, consider the SCARA manipulator of Figure 3.11. This manipulator, which is an abstraction of the AdeptOne robot of Figure 1.11, consists of an RRP arm and a one degree-of-freedom wrist, whose motion is a roll about the vertical axis. The first step is to locate and label the joint axes as shown. Since all joint axes are parallel we have some freedom in the placement of the origins. The origins are placed as shown for convenience. We establish the x_0 axis in the plane of the page as shown. This is completely arbitrary and only affects the zero configuration of the manipulator, that is, the position of the manipulator when $\theta_1 = 0$.

The joint parameters are given in Table 3.5, and the A-matrices are as follows.

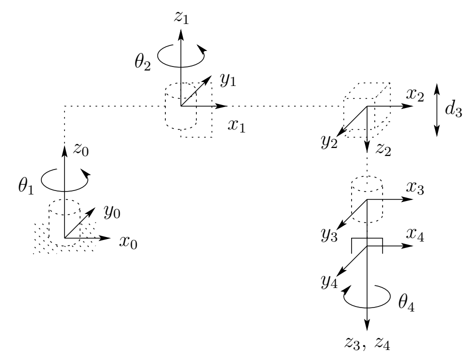

Figure 3.11: DH coordinate frame assignment for the SCARA manipulator.

Table 3.5: Joint parameters for SCARA.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	⋆
2	a_2	180	0	⋆
3	0	0	⋆	0
4	0	0	d_4	⋆

* joint variable

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.45}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.46}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.47}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.48}$$

The forward kinematic equations are therefore given by

$$T_4^0 = A_1 \cdots A_4 \quad = \quad \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.49)$$

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Chapter 4

INVERSE KINEMATICS

In the previous chapter we showed how to determine the end-effector position and orientation in terms of the joint variables. This chapter is concerned with the inverse problem of finding the joint variables in terms of the end-effector position and orientation. This is the problem of **inverse kinematics**, and it is, in general, more difficult than the forward kinematics problem.

In this chapter, we begin by formulating the general inverse kinematics problem. Following this, we describe the principle of kinematic decoupling and how it can be used to simplify the inverse kinematics of most modern manipulators. Using kinematic decoupling, we can consider the position and orientation problems independently. We describe a geometric approach for solving the positioning problem, while we exploit the Euler angle parameterization to solve the orientation problem.

4.1 The General Inverse Kinematics Problem

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) \tag{4.1}$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \ldots, q_n) = H \tag{4.2}$$

where

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n). \tag{4.3}$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables $q_1, \ldots, q_n$ so that $T_n^0(q_1, \ldots, q_n) = H$.

Equation (4.2) results in twelve nonlinear equations in n unknown variables, which can be written as

$$T_{ij}(q_1, \ldots, q_n) = h_{ij}, \qquad i = 1, 2, 3, \quad j = 1, \ldots, 4 \tag{4.4}$$

where T_{ij}, h_{ij} refer to the twelve nontrivial entries of T_n^0 and H, respectively. (Since the bottom row of both T_n^0 and H are (0,0,0,1), four of the sixteen equations represented by (4.2) are trivial.)

Example 4.1

Recall the Stanford manipulator of Example 3.3.5. Suppose that the desired position and orientation of the final frame are given by

$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations (cf. (3.43) and (3.44)):

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= r_{11} \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= r_{21} \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5s_6 &= r_{31} \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= r_{12} \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= r_{22} \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= r_{32} \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= r_{13} \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= r_{23} \\
-s_2c_4s_5 + c_2c_5 &= r_{33} \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= o_x \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= o_y \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= o_z.
\end{aligned}$$

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The equations in the preceding example are, of course, much too difficult to solve directly in closed form. This is the case for most robot arms. Therefore, we need to develop efficient and systematic techniques that exploit the particular kinematic structure of the manipulator. Whereas the forward kinematics problem always has a unique solution that can be obtained simply by evaluating the forward equations, the inverse kinematics problem may or may not have a solution. Even if a solution exists, it may or may not be unique.

Furthermore, because these forward kinematic equations are in general complicated nonlinear functions of the joint variables, the solutions may be difficult to obtain even when they exist.

In solving the inverse kinematics problem we are most interested in finding a closed form solution of the equations rather than a numerical solution. Finding a closed form solution means finding an explicit relationship:

$$q_k = f_k(h_{11}, \ldots, h_{34}), \qquad k = 1, \ldots, n. \tag{4.6}$$

Closed form solutions are preferable for two reasons. First, in certain applications, such as tracking a welding seam whose location is provided by a vision system, the inverse kinematic equations must be solved at a rapid rate, say every 20 milliseconds, and having closed form expressions rather than an iterative search is a practical necessity. Second, the kinematic equations in general have multiple solutions. Having closed form solutions allows one to develop rules for choosing a particular solution among several.

The practical question of the existence of solutions to the inverse kinematics problem depends on engineering as well as mathematical considerations. For example, the motion of the revolute joints may be restricted to less than a full 360 degrees of rotation so that not all mathematical solutions of the kinematic equations will correspond to physically realizable configurations of the manipulator. We will assume that the given position and orientation is such that at least one solution of (4.2) exists. Once a solution to the mathematical equations is identified, it must be further checked to see whether or not it satisfies all constraints on the ranges of possible joint motions. For our purposes here we henceforth assume that the given homogeneous matrix H in (4.2) corresponds to a configuration within the manipulator's workspace with an attainable orientation. This then guarantees that the mathematical solutions obtained correspond to achievable configurations.

4.2 Kinematic Decoupling

Although the general problem of inverse kinematics is quite difficult, it turns out that for manipulators having six joints, with the last three joints intersecting at a point (such as the Stanford Manipulator above), it is possible to decouple the inverse kinematics problem into two simpler problems, known respectively, as **inverse position kinematics**, and **inverse orientation kinematics**. To put it another way, for a six-DOF manipulator with a spherical wrist, the inverse kinematics problem may be separated into two simpler problems, namely first finding the position of the intersection of the wrist axes, hereafter called the **wrist center**, and then finding the orientation of the wrist.

For concreteness let us suppose that there are exactly six degrees-of-freedom and that the last three joint axes intersect at a point o_c. We express (4.2) as two sets of equations representing the rotational and positional equations

$$R_6^0(q_1, \ldots, q_6) = R \tag{4.7}$$

$$o_6^0(q_1, \ldots, q_6) = o \tag{4.8}$$

where o and R are the desired position and orientation of the tool frame, expressed with respect to the world coordinate system. Thus, we are given o and R, and the inverse kinematics problem is to solve for $q_1, \ldots, q_6$.

The assumption of a spherical wrist means that the axes z_3, z_4, and z_5 intersect at o_c and hence the origins o_4 and o_5 assigned by the DH-convention will always be at the wrist center o_c. Often o_3 will also be at o_c, but this is not necessary for our subsequent development. The important point of this assumption for the inverse kinematics is that motion of the final three links about these axes will not change the position of o_c, and thus, the position of the wrist center is thus a function of only the first three joint variables.

The origin of the tool frame (whose desired coordinates are given by o) is simply obtained by a translation of distance d_6 along z_5 from o_c (see Table 3.3). In our case, z_5 and z_6 are the same axis, and the third column of R expresses the direction of z_6 with respect to the base frame. Therefore, we have

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.9}$$

Thus in order to have the end-effector of the robot at the point with coordinates given by o and with the orientation of the end-effector given by $R = (r_{ij})$, it is necessary and sufficient that the wrist center o_c have coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.10}$$

and that the orientation of the frame $o_6x_6y_6z_6$ with respect to the base be given by R. If the components of the end-effector position o are denoted o_x, o_y, o_z and the components of the wrist center o_c^0 are denoted x_c, y_c, z_c then (4.10) gives the relationship

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}. \tag{4.11}$$

Using Equation (4.11) we may find the values of the first three joint variables. This determines the orientation transformation R_3^0 which depends only on these first three joint variables. We can now determine the orientation of the end-effector relative to the frame $o_3x_3y_3z_3$ from the expression

$$R = R_3^0 R_6^3 \tag{4.12}$$

as

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R. \tag{4.13}$$

As we shall see in Section 4.4, the final three joint angles can then be found as a set of Euler angles corresponding to R_6^3. Note that the right hand side of (4.13) is completely known since R is given and R_3^0 can be calculated once the first three joint variables are known. The idea of kinematic decoupling is illustrated in Figure 4.1.

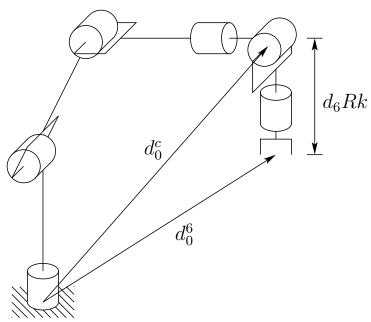

Figure 4.1: Kinematic decoupling.

Summary

For this class of manipulators the determination of the inverse kinematics can be summarized by the following algorithm.

Step 1: Find q_1, q_2, q_3 such that the wrist center o_c has coordinates given by

$$o_c^0 \quad = \quad o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.14}$$

Step 2: Using the joint variables determined in Step 1, evaluate R_3^0.

Step 3: Find a set of Euler angles corresponding to the rotation matrix

$$R_6^3 \quad = \quad (R_3^0)^{-1} R = (R_3^0)^T R. \tag{4.15}$$

4.3 Inverse Position: A Geometric Approach

For the common kinematic arrangements that we consider, we can use a geometric approach to find the variables, q_1, q_2, q_3 corresponding to o_c^0 given by (4.10). We restrict our treatment to the geometric approach for two reasons. First, as we have said, most present manipulator designs are kinematically simple, usually consisting of one of the five basic configurations of Chapter 1 with a spherical wrist. Indeed, it is partly due to the difficulty of the general inverse kinematics problem that manipulator designs have evolved to their present state. Second, there are few techniques that can handle the general inverse kinematics problem for arbitrary configurations. Since the reader is most likely to encounter robot configurations

of the type considered here, the added difficulty involved in treating the general case seems unjustified. The reader is directed to the references at the end of the chapter for treatment of the general case.

In general the complexity of the inverse kinematics problem increases with the number of nonzero link parameters. For most manipulators, many of the a_i, d_i are zero, the α_i are 0 or $\pm\pi/2$, etc. In these cases especially, a geometric approach is the simplest and most natural. We will illustrate this with several important examples.

Articulated Configuration

Consider the elbow manipulator shown in Figure 4.2, with the components of o_c^0 denoted by x_c, y_c, z_c. We project o_c onto the $x_0 - y_0$ plane as shown in Figure 4.3.

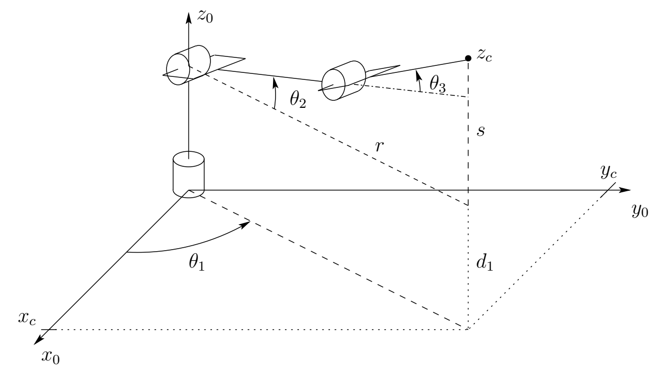

Figure 4.2: Elbow manipulator.

We see from this projection that

$$\theta_1 \quad = \quad A\tan(x_c, y_c), \tag{4.16}$$

in which $A\tan(x, y)$ denotes the two argument arctangent function. $A\tan(x, y)$ is defined for all $(x, y) \neq (0, 0)$ and equals the unique angle θ such that

$$\cos\theta = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}, \qquad \sin\theta = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}. \tag{4.17}$$

For example, $A\tan(1, -1) = -\frac{\pi}{4}$, while $A\tan(-1, 1) = +\frac{3\pi}{4}$.

Note that a second valid solution for θ_1 is

$$\theta_1 \quad = \quad \pi + A\tan(x_c, y_c). \tag{4.18}$$

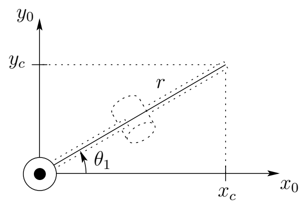

Figure 4.3: Projection of the wrist center onto $x_0 - y_0$ plane.

Of course this will, in turn, lead to different solutions for θ_2 and θ_3, as we will see below.

These solutions for θ_1, are valid unless $x_c = y_c = 0$. In this case (4.16) is undefined and the manipulator is in a singular configuration, shown in Figure 4.4. In this position the

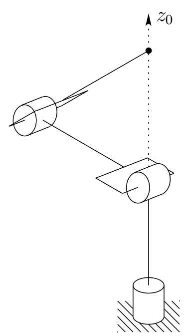

Figure 4.4: Singular configuration.

wrist center o_c intersects z_0; hence any value of θ_1 leaves o_c fixed. There are thus infinitely many solutions for θ_1 when o_c intersects z_0.

If there is an offset $d \neq 0$ as shown in Figure 4.5 then the wrist center cannot intersect z_0. In this case, depending on how the DH parameters have been assigned, we will have $d_2 = d$ or $d_3 = d$. In this case, there will, in general, be only two solutions for θ_1. These correspond to the so-called **left arm** and **right arm** configurations as shown in Figures 4.6 and 4.7. Figure 4.6 shows the left arm configuration. From this figure, we see geometrically that

$$\theta_1 = \phi - \alpha \tag{4.19}$$

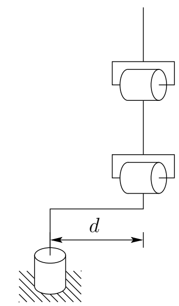

Figure 4.5: Elbow manipulator with shoulder offset.

where

$$\phi = A\tan(x_c, y_c) \tag{4.20}$$

$$\alpha = A\tan\left(\sqrt{r^2 - d^2}, d\right) \tag{4.21}$$

$$= A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right).$$

The second solution, given by the right arm configuration shown in Figure 4.7 is given by

$$\theta_1 = A\tan(x_c, y_c) + A\tan\left(-\sqrt{r^2 - d^2}, -d\right). \tag{4.22}$$

To see this, note that

$$\theta_1 = \alpha + \beta \tag{4.23}$$

$$\alpha = A\tan(x_c, y_c) \tag{4.24}$$

$$\beta = \gamma + \pi \tag{4.25}$$

$$\gamma = A\tan(\sqrt{r^2 - d^2}, d) \tag{4.26}$$

$$\tag{4.27}$$

which together imply that

$$\beta = A\tan\left(-\sqrt{r^2 - d^2}, -d\right) \tag{4.28}$$

since $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$.

To find the angles θ_2, θ_3 for the elbow manipulator, given θ_1, we consider the plane formed by the second and third links as shown in Figure 4.8. Since the motion of links

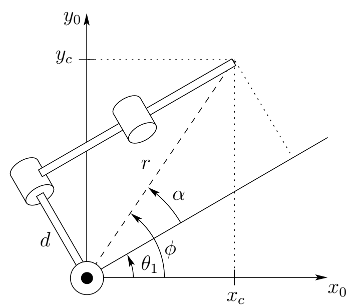

Figure 4.6: Left arm configuration.

two and three is planar, the solution is analogous to that of the two-link manipulator of Chapter 1. As in our previous derivation (cf. (1.8) and (1.9)) we can apply the law of cosines to obtain

$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3} \tag{4.29}$$
$$= \frac{x_c^2 + y_c^2 - d^2 + z_c^2 - a_2^2 - a_3^2}{2a_2a_3} := D,$$

since $r^2 = x_c^2 + y_c^2 - d^2$ and $s = z_c$. Hence, θ_3 is given by

$$\theta_3 = A\tan\left(D, \pm\sqrt{1 - D^2}\right). \tag{4.30}$$

Similarly θ_2 is given as

$$\theta_2 = A\tan(r, s) - A\tan(a_2 + a_3c_3, a_3s_3) \tag{4.31}$$
$$= A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c\right) - A\tan(a_2 + a_3c_3, a_3s_3).$$

The two solutions for θ_3 correspond to the elbow-up position and elbow-down position, respectively.

An example of an elbow manipulator with offsets is the PUMA shown in Figure 4.9. There are four solutions to the inverse position kinematics as shown. These correspond to the situations left arm-elbow up, left arm–elbow down, right arm–elbow up and right arm–elbow down. We will see that there are two solutions for the wrist orientation thus giving a total of eight solutions of the inverse kinematics for the PUMA manipulator.

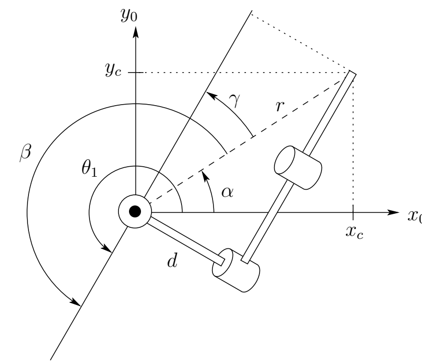

Figure 4.7: Right arm configuration.

Spherical Configuration

We next solve the inverse position kinematics for a three degree of freedom spherical manipulator shown in Figure 4.10. As in the case of the elbow manipulator the first joint variable is the base rotation and a solution is given as

$$\theta_1 = A\tan(x_c, y_c) \tag{4.32}$$

provided x_c and y_c are not both zero. If both x_c and y_c are zero, the configuration is singular as before and θ_1 may take on any value.

The angle θ_2 is given from Figure 4.10 as

$$\theta_2 = A\tan(r, s) + \frac{\pi}{2} \tag{4.33}$$

where $r^2 = x_c^2 + y_c^2$, $s = z_c - d_1$. As in the case of the elbow manipulator a second solution for θ_1 is given by

$$\theta_1 = \pi + A\tan(x_c, y_c); \tag{4.34}$$

The linear distance d_3 is found as

$$d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}. \tag{4.35}$$

The negative square root solution for d_3 is disregarded and thus in this case we obtain two solutions to the inverse position kinematics as long as the wrist center does not intersect z_0. If there is an offset then there will be left and right arm configurations as in the case of the elbow manipulator (Problem 4-12).

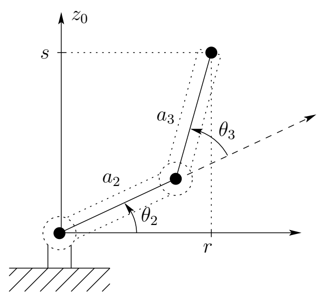

Figure 4.8: Projecting onto the plane formed by links 2 and 3.

4.4 Inverse Orientation

In the previous section we used a geometric approach to solve the inverse position problem. This gives the values of the first three joint variables corresponding to a given position of the wrist origin. The inverse orientation problem is now one of finding the values of the final three joint variables corresponding to a given orientation with respect to the frame $o_3x_3y_3z_3$. For a spherical wrist, this can be interpreted as the problem of finding a set of Euler angles corresponding to a given rotation matrix R. Recall that equation (3.32) shows that the rotation matrix obtained for the spherical wrist has the same form as the rotation matrix for the Euler transformation, given in (2.52). Therefore, we can use the method developed in Section 2.5.1 to solve for the three joint angles of the spherical wrist. In particular, we solve for the three Euler angles, ϕ, θ, ψ, using Equations (2.54) – (2.59), and then use the mapping

$$\begin{aligned} \theta_4 &= \phi, \\ \theta_5 &= \theta, \\ \theta_6 &= \psi. \end{aligned}$$

Example 4.2 Articulated Manipulator with Spherical Wrist
The DH parameters for the frame assignment shown in Figure 4.2 are summarized in Table 4.1. Multiplying the corresponding A_i matrices gives the matrix R_3^0 for the articulated or elbow manipulator as

$$R_3^0 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}. \quad (4.36)$$

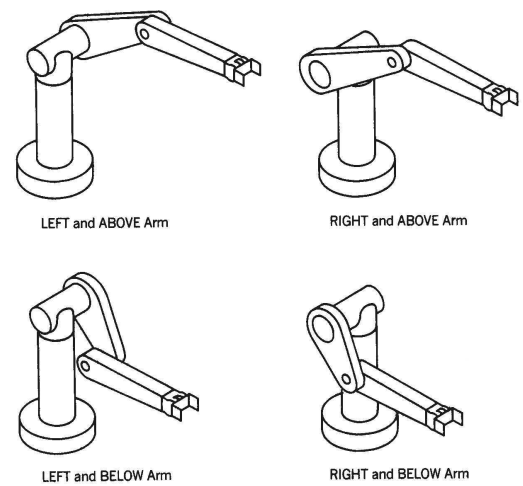

Figure 4.9: Four solutions of the inverse position kinematics for the PUMA manipulator.

Table 4.1: Link parameters for the articulated manipulator of Figure 4.2.

Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*

* variable

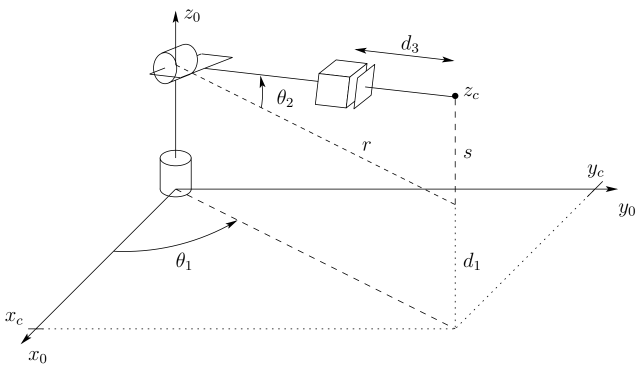

Figure 4.10: Spherical manipulator.

The matrix $R_6^3 = A_4 A_5 A_6$ *is given as*

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}. \quad (4.37)$$

The equation to be solved now for the final three variables is therefore

$$R_6^3 = (R_3^0)^T R \quad (4.38)$$

and the Euler angle solution can be applied to this equation. For example, the three equations given by the third column in the above matrix equation are given by

$$c_4s_5 = c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33} \quad (4.39)$$
$$s_4s_5 = -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33} \quad (4.40)$$
$$c_5 = s_1r_{13} - c_1r_{23}. \quad (4.41)$$

Hence, if not both of the expressions (4.39), (4.40) are zero, then we obtain θ_5 *from (2.54) and (2.55) as*

$$\theta_5 = A\tan\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right). \quad (4.42)$$

If the positive square root is chosen in (4.42), then θ_4 *and* θ_6 *are given by (2.56) and (2.57), respectively, as*

$$\theta_4 = A\tan(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33},$$
$$-c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \quad (4.43)$$
$$\theta_6 = A\tan(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}). \quad (4.44)$$

The other solutions are obtained analogously. If $s_5 = 0$, *then joint axes* z_3 *and* z_5 *are collinear. This is a singular configuration and only the sum* $\theta_4 + \theta_6$ *can be determined. One solution is to choose* θ_4 *arbitrarily and then determine* θ_6 *using (2.62) or (2.64).*
◇

Example 4.3 Summary of Elbow Manipulator Solution

To summarize the preceding development we write down one solution to the inverse kinematics of the six degree-of-freedom elbow manipulator shown in Figure 4.2 which has no joint offsets and a spherical wrist.

Given

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}; \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (4.45)$$

then with

$$x_c = o_x - d_6r_{13} \quad (4.46)$$
$$y_c = o_y - d_6r_{23} \quad (4.47)$$
$$z_c = o_z - d_6r_{33} \quad (4.48)$$

a set of D-H joint variables is given by

$$\theta_1 = A\tan(x_c, y_c) \quad (4.49)$$
$$\theta_2 = A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c\right) - A\tan(a_2 + a_3c_3, a_3s_3) \quad (4.50)$$
$$\theta_3 = A\tan\left(D, \pm\sqrt{1 - D^2}\right),$$
$$\text{where } D = \frac{x_c^2 + y_c^2 - d^2 + z_c^2 - a_2^2 - a_3^2}{2a_2a_3} \quad (4.51)$$
$$\theta_4 = A\tan(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33},$$
$$-c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \quad (4.52)$$
$$\theta_5 = A\tan\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right). \quad (4.53)$$
$$\theta_6 = A\tan(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}). \quad (4.54)$$

The other possible solutions are left as an exercise (Problem 4-11). ◇

Example 4.4 SCARA Manipulator

As another example, we consider the SCARA manipulator whose forward kinematics is defined by T_4^0 *from (3.49). The inverse kinematics is then given as the set of solutions of the equation*

$$\begin{bmatrix} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}. \quad (4.55)$$

We first note that, since the SCARA has only four degrees-of-freedom, not every possible H from $SE(3)$ allows a solution of (4.55). In fact we can easily see that there is no solution of (4.55) unless R is of the form

$$R = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ s_\alpha & -c_\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{4.56}$$

and if this is the case, the sum $\theta_1 + \theta_2 - \theta_4$ is determined by

$$\theta_1 + \theta_2 - \theta_4 = \alpha = A\tan(r_{11}, r_{12}). \tag{4.57}$$

Projecting the manipulator configuration onto the $x_0 - y_0$ plane immediately yields the situation of Figure 4.11.

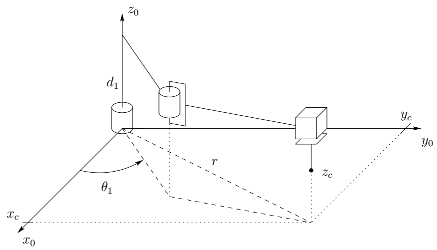

Figure 4.11: SCARA manipulator.

We see from this that

$$\theta_2 = A\tan\left(c_2, \pm\sqrt{1-c_2}\right) \tag{4.58}$$

where

$$c_2 = \frac{o_x^2 + o_y^2 - a_1^2 - a_2^2}{2a_1a_2} \tag{4.59}$$

$$\theta_1 = A\tan(o_x, o_y) - A\tan(a_1 + a_2c_2, a_2s_2). \tag{4.60}$$

We may then determine θ_4 from (4.57) as

$$\begin{aligned} \theta_4 &= \theta_1 + \theta_2 - \alpha \\ &= \theta_1 + \theta_2 - A\tan(r_{11}, r_{12}). \end{aligned} \tag{4.61}$$

Finally d_3 is given as

$$d_3 = o_z + d_4. \tag{4.62}$$

◇