# Computer Vision
# Projective Geometry and Calibration

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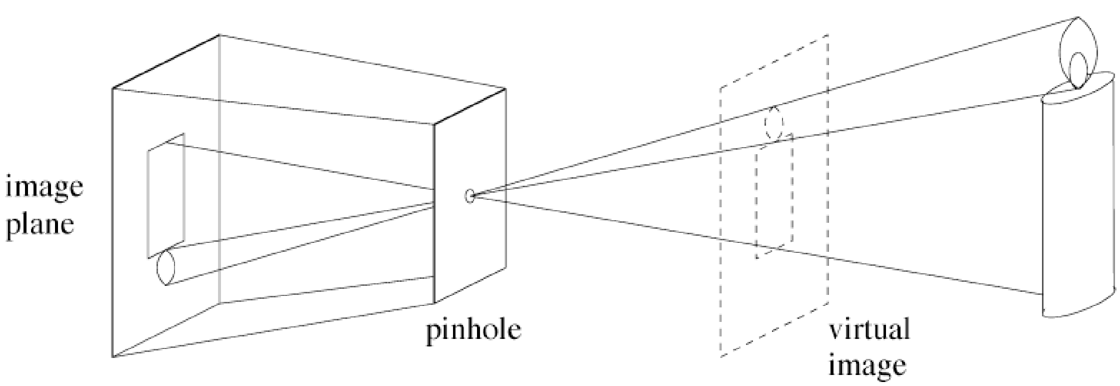
# Pinhole cameras

- Abstract camera model - box with a small hole in it
- Pinhole cameras work in practice

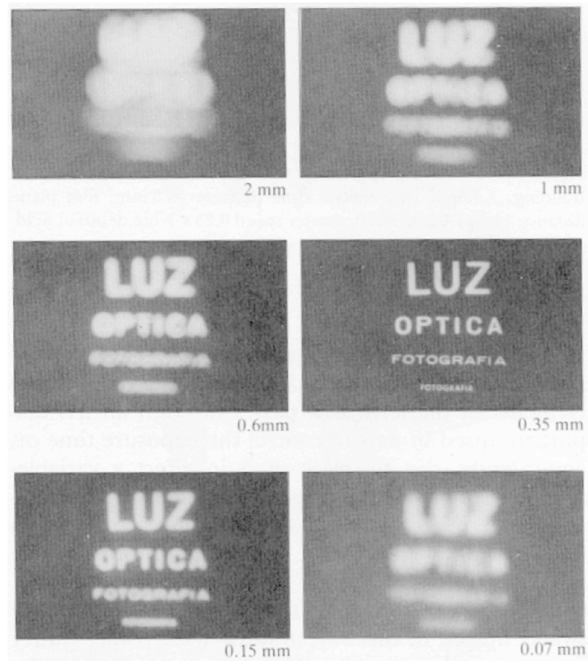


# Real Pinhole Cameras

Pinhole too big -
many directions are averaged, blurring the image

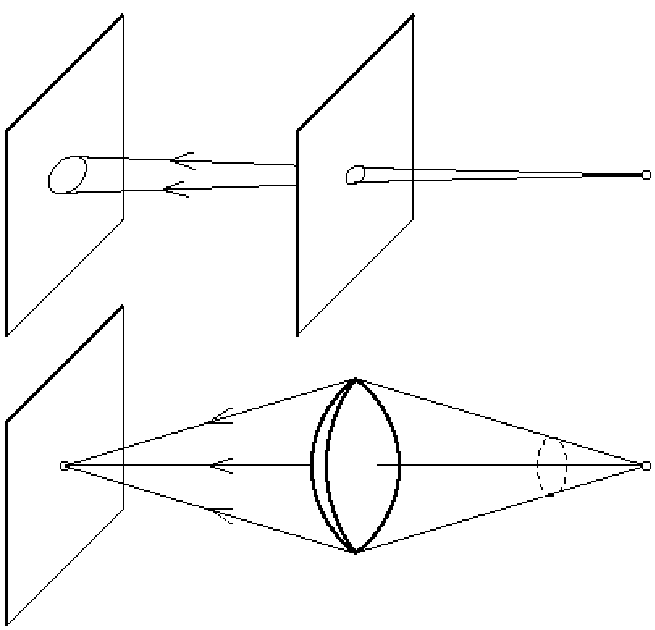
Pinhole too small-
diffraction effects blur the image

Generally, pinhole cameras are *dark*, because a very small set of rays from a particular point hits the screen.



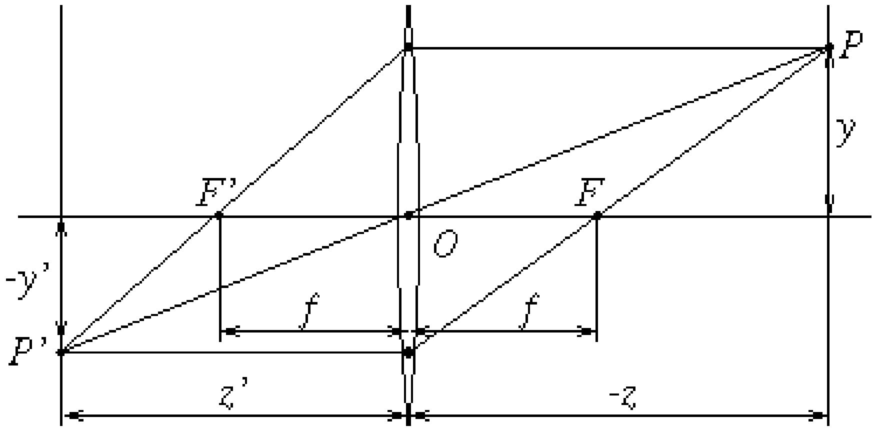



# The reason for lenses



Lenses gather and focus light, allowing for brighter images.

# The thin lens



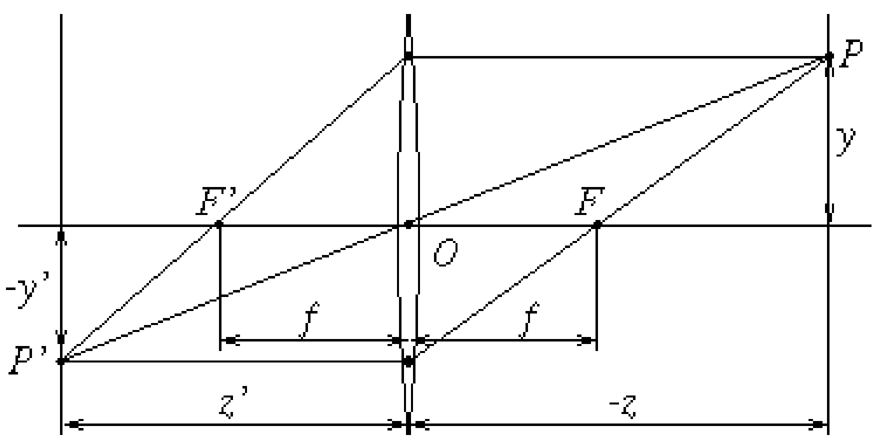

Thin Lens Properties:

1. A ray entering parallel to optical axis goes through the focal point.
2. A ray emerging from focal point is parallel to optical axis
3. A ray through the optical center is unaltered

$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}$$



# The thin lens




$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}$$

Note that, if the image plane is very small and/or z >> z', then z' is about f

# Field of View

- The *effective diameter* of a lens (d) is the portion of a lens actually reachable by light rays.
- The effective diameter and the focal length determine the field of view:

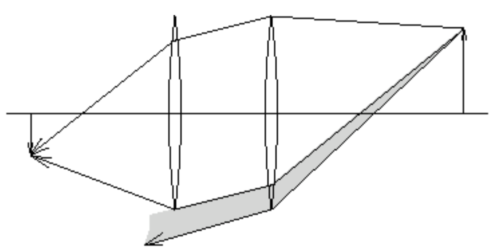
$$\tan w = d/(2f)$$

- w is the half the total angular "view" of a lens system.
- Another fact is that in practice points at different distances are imaged, leading to so-called "circles of confusion" of size d/z | z'-z| where z is the nominal image plane and z' is the focusing distance given by the thin lens equation.
- The "depth of field" is the range of distances that produce acceptably focused images. Depth of field varies inversely with focal length and lens diameter.



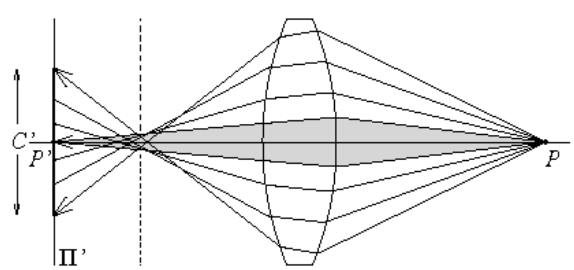
# Lens Realities

Real lenses have a finite depth of field, and usually suffer from a variety of defects



vignetting

Spherical Aberration



# Standard Camera Coordinates

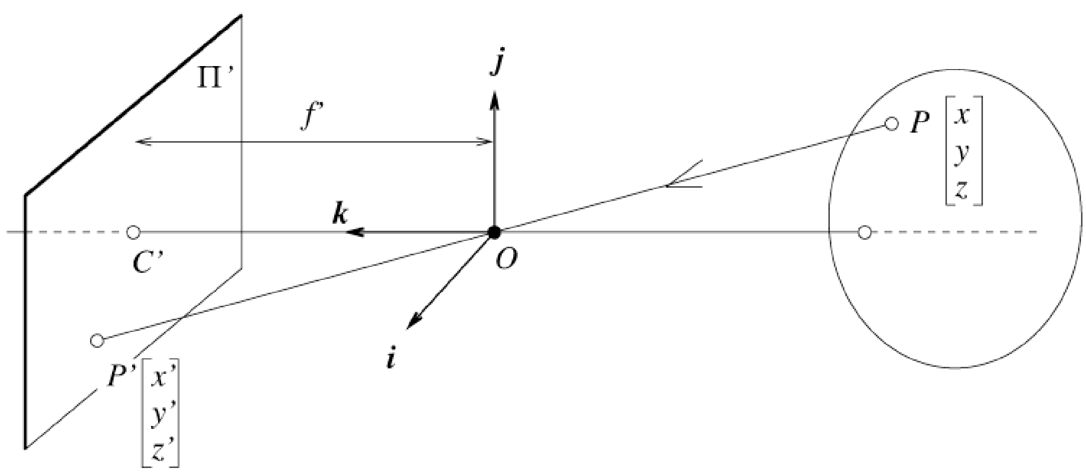
- By convention, we place the image in front of the optical center
  - typically we approximate by saying it lies one focal distance from the center
  - in reality this can't be true for a finite size chip!
- Optical axis is z axis pointing outward
- X axis is parallel to the scanlines (rows) pointing to the right!
- By the right hand rule, the Y axis must point downward
- Note this corresponds with indexing an image from the upper left to the lower right, where the X coordinate is the column index and the Y coordinate is the row index.



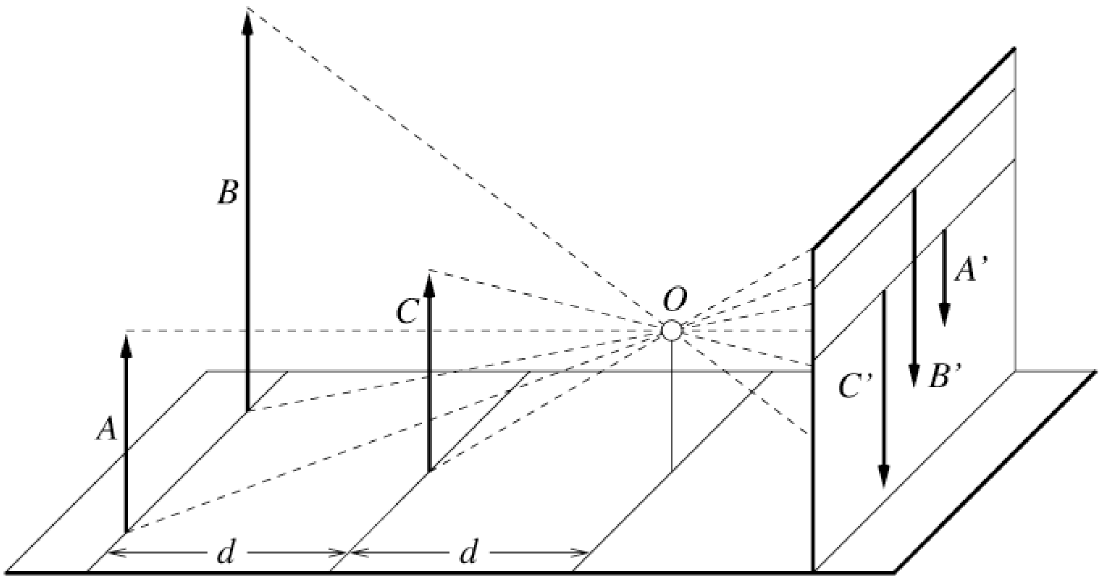
# The equation of projection

- Equating z' and f
  - We have, by similar triangles, that (x, y, z) -> (-f x/z, -f y/z, -f)
  - Ignore the third coordinate, and flip the image around to get:

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$



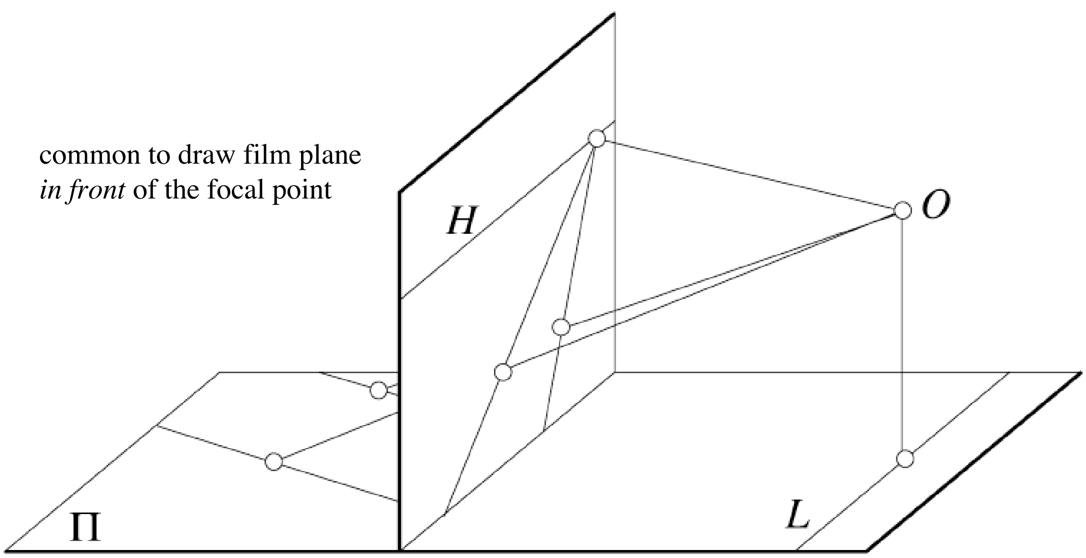
# Distant objects are smaller






# Parallel lines meet

common to draw film plane
*in front* of the focal point




A Good Exercise: Show this is the case!

# Some Useful Geometry

- In 3D space
  - points:
    - Cartesian point (x,y,z)
    - Projective pt (x,y,z,w) with convention that w is a scale factor
  - lines:
    - a point p on the line and unit vector v for direction
      - for minimal parameterization, p is closest point to origin
    - Alternative, a line is the intersection of two planes (see below)
  - planes
    - a point p on the plane and a unit *normal* n s.t. n . (p' - p) = 0
    - multiplying through, also n.p' - d = 0, where d is distance of closest pt to origin.
    - any vector n . q = 0 where q is a projective pt
      - note, for two planes, the intersection is two equations in 4 unknowns up to scale --- i.e. a one-dimensional subspace, or a *line*
    - Note that planes and points are *dual* --- in the above, I can equally think of n or q as the normal (resp. point).



# Some Useful Geometry

- In 2D space
  - points:
    - Cartesian point (x,y)
    - Projective pt (x,y,w) with convention that w is a scale factor
  - lines
    - a point p on the line and a unit *normal* n s.t. n . (p' - p) = 0
    - multiplying through, also n.p' - d = 0, where d is distance of closest pt to origin.
    - any vector n . q = 0 where q is a projective pt
      - note, for two lines, the intersection is two equations in 3 unknowns up to scale --- i.e. a one-dimensional subspace, or a *point*
    - note that points and lines are *dual* --- I can think of n or q as the normal (resp. point)

# Some Projective Concepts

- The vector p = (x,y,z,w)' is equivalent to the vector k p for nonzero k
  - note the vector p = 0 is disallowed from this representation
- The vector v = (x,y,z,0)' is termed a "point at infinity"; it corresponds to a direction
- In $P^2$,
  - given two points $p_1$ and $p_2$, l = $p_1$ £ $p_2$ is the line containing them
  - given two lines, $l_1$, and $l_2$, p = $l_1$ £ $l_2$ is point of intersection
  - A point p lies on a line l if p ¢ l = 0 (note this is a consequence of the triple product rule)
  - l = (0,0,1) is the "line at infinity"
  - it follows that, for any point p at infinity, l¢ p = 0, which implies that points at infinity lie on the line at infinity.



# Some Projective Concepts

- The vector p = (x,y,z,w)' is equivalent to the vector k p for nonzero k
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- In $P^3$,
  - A point p lies on a plane l if p ¢ l = 0 (note this is a consequence of the triple product rule; there is an equivalent expression in determinants)
  - l = (0,0,0,1) is the "plane at infinity"
  - it follows that, for any point p at infinity, l¢ p = 0, which implies that points at infinity lie on the line at infinity.

# Some Projective Concepts

- The vector p = (x,y,z,w)' is equivalent to the vector k p for nonzero k
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- The vector v = (x,y,z,0)' is termed a "point at infinity"; it corresponds to a direction
- Plucker coordinates
  - In general, a representation for a line through points $p_1$ and $p_2$ is given by all possible 2x2 determinants of $[p_1\ p_2]$ (an n by 2 matrix)
    - $u = (l_{4,1}, l_{4,1}, l_{4,3}, l_{2,3}, l_{3,1}, l_{1,2})$ are the Pl_cker coordinates of the line passing through the two points.
    - if the points are not at infinity, then this is also the same as $(\underline{p}_2 - \underline{p}_1, \underline{p}_1 \times \underline{p}_2)$
  - The first 3 coordinates are the direction of the line
  - The second 3 are the normal to the plane (in $R^3$) containing the origin and the points
  - In general, a representation for a plane passing through three points $p_1$, $p_2$ and $p_3$ are the determinants of all 3 by 3 submatrices $[p_1\ p_2\ p_3]$
    - let $l_{i,j}$ mean the determinant of the matrix of matrix formed by the rows i and j
    - $P = (l_{234}, l_{134}, l_{142}, l_{123})$
    - Note the three points are colinear if all four of these values are zero (hence the original 3x4 matrix has rank 2, as we would expect).
  - Two lines are colinear if we create the 4x4 matrix $[p_1, p_2, p'_1, p'_2]$ where the p's come from one line, and the p's come from another.



# Parallel lines meet

- First, show how lines project to images.
- Second, consider lines that have the same direction (are parallel)
- Third, consider the degenerate case of lines parallel in the image
  - (by convention, the vanishing point is at infinity!)

A Good Exercise: Show this is the case!

# Vanishing points

- Another good exercise (really follows from the previous one): show the form of projection of *lines* into images.

- Each set of parallel lines (=direction) meets at a different point
  - The *vanishing point* for this direction

- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane



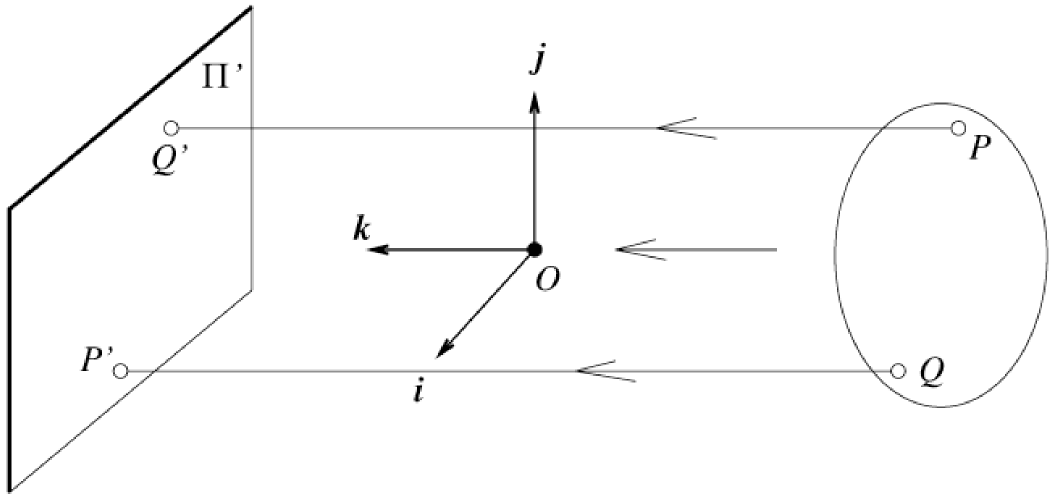
# The Camera Matrix

- Homogenous coordinates for 3D
  - four coordinates for 3D point
  - equivalence relation (X,Y,Z,T) is the same as (k X, k Y, k Z,k T)
- Turn previous expression into HC's
  - HC's for 3D point are (X,Y,Z,T)
  - HC's for point in image are (U,V,W)

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

$$(U, V, W) \rightarrow (\frac{U}{W}, \frac{V}{W}) = (u, v)$$

# Orthographic projection




Suppose I let f go to infinity; then

$$u = x$$

$$v = y$$



# The model for orthographic projection

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

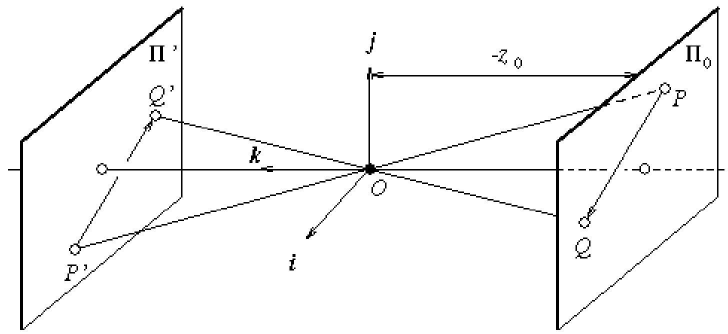
# Weak perspective

- Issue
  - perspective effects, but not over the scale of individual objects
  - collect points into a group at about the same depth, then divide each point by the depth of its group
  - Adv: easy
  - Disadv: wrong

$$u = sx$$

$$v = sy$$

$$s = f / Z^*$$






# The model for weak perspective projection

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & Z^*/f \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

# The Affine Camera

- Choose a nominal point $x_0$, $y_0$, $z_0$ and describe projection relative to that point

- $u = f\,[x_0/z_0 + (x-x_0)/z_0 - x_0/z^2_0\,(z - z_0)\,] = f\,(a_1\,x + a_2\,z + d_1)$
- $v = f\,[y_0/z_0 + (y - y_0)/z_0 - y_0/z^2_0\,(z - z_0) = f\,(a_3\,y + a_4\,z + d_2)$

- gathering up

alternatively:

- $A = [a_1\ 0\ a_2;\ 0\ a_3\ a_4]$
- $d = [d_1;\ d_2]$

- $\mathbf{u} = A\,\mathbf{P} + d$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} a_1 & 0 & a_2 & d_1 \\ 0 & a_3 & a_4 & d_2 \\ 0 & 0 & 0 & 1/f \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$



# Geometric Transforms

In general, a point in n-D space transforms by

P' = rotate(point) + translate(point)

In 2-D space, this can be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

In 3-D space (or n-D), this can generalized as a matrix equation:

$p' = R\,p + T$ or $p = R^t\,(p' - T)$

# Geometric Transforms

Now, using the idea of homogeneous transforms, we can write:

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

R and T both require 3 parameters. These correspond to the 6 *extrinsic parameters* needed for camera calibration



# Intrinsic Parameters

*Intrinsic Parameters* describe the conversion from unit focal length metric to pixel coordinates (and the reverse)

$x_{mm} = - (x_{pix} – o_x) s_x$ --> $-1/s_x\, x_{mm} + o_x = x_{pix}$
$y_{mm} = - (y_{pix} – o_y) s_y$ --> $-1/s_y\, y_{mm} + o_y = y_{pix}$

or

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix}_{pix} = \begin{pmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}_{mm} = K_{\mathrm{int}}\, p$$

It is common to combine scale and focal length together as the are both scaling factors; note projection is unitless in this case!

# The Camera Matrix

- Homogenous coordinates for 3D
  - four coordinates for 3D point
  - equivalence relation (X,Y,Z,T) is the same as (k X, k Y, k Z,k T)
- Turn previous expression into HC's
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$$(U,V,W) \rightarrow (\frac{U}{W}, \frac{V}{W}) = (u,v)$$



# Camera parameters

- Summary:
  - points expressed in external frame
  - points are converted to canonical camera coordinates
  - points are projected
  - points are converted to pixel units

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{intrinsic parameters} \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{projection model} \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{extrinsic parameters} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

point in pixel coords.  point in metric image coords.  point in cam. coords.  point in world coords.

# Lens Distortion

- In general, lens introduce minor irregularities into images, typically radial distortions:

$$x = x_d(1 + k_1 r^2 + k_2 r^4)$$
$$y = y_d(1 + k_1 r^2 + k_2 r^4)$$
$$r^2 = x_d^2 + y_d^2$$

- The values $k_1$ and $k_2$ are additional parameters that must be estimated in order to have a model for the camera system.



# Summary: Other Models

- The *orthographic* and *scaled orthographic* cameras (also called *weak perspective*)
  - simply ignore z
  - differ in the scaling from x/y to u/v coordinates
  - preserve Euclidean structure to a great degree

- The *affine camera* is a generalization of orthographic models.
  - u = A p + d
  - A is 2 x 3 and d is 2x1
  - This can be derived from scaled orthography or by linearizing perspective about a point not on the optical axis

- The *projective camera* is a generalization of the perspective camera.
  - u' = M p
  - M is 3x4 nonsingular defined up to a scale factor
  - This just a generalization (by one parameter) from “real” model

- Both have the advantage of being linear models on real and projective spaces, respectively.

# Related Transformation Models

- Euclidean models (homogeneous transforms); ${}^b p = {}^b T_a \, {}^a p$
- Similarity models: ${}^b p = s \, {}^b T_a \, {}^a p$
- Affine models: ${}^b p = {}^b K_a \, {}^a p$, K = [A,t;0 0 0 1], A 2 GL(3)
- Projective models: ${}^b p = {}^b M_a \, {}^a p$, M 2 GL(4)
  - Ray models
  - Affine plane
  - Sphere



# Model Stratification

| | Euclidean | Similarity | Affine | Projective |
|---|---|---|---|---|
| Transforms | | | | |
| rotation | x | x | x | x |
| translation | x | x | x | x |
| uniform scaling | | x | x | x |
| nonuniform scaling | | | x | x |
| shear | | | x | x |
| perspective | | | | x |
| composition of proj. | | | | x |
| Invariants | | | | |
| length | x | | | |
| angle | x | x | | |
| ratios | x | x | | |
| parallelism | x | x | x | |
| incidence/cross rat. | x | x | x | x |

# Why Projective (or Affine or ...)

- Recall in Euclidean space, we can define a change of coordinates by choosing a new origin and three orthogonal unit vectors that are the new coordinate axes
  – The class of all such transformation is SE(3) which forms a group
  – One rendering is the class of all homogeneous transformations
  – This *does not* model what happens when things are imaged (why?)
- If we allow a change in scale, we arrive at similarity transforms, also a group
  – This sometimes can model what happens in imaging (when?)
- If we allow the 3x3 rotation to be an arbitrary member of GL(3) we arrive at affine transformations (yet another group!)
  – This also sometimes is a good model of imaging
  – The basis is now defined by three arbitrary, non-parallel vectors
- The process of perspective projection **does not** form a group
  – that is, a picture of a picture cannot in general be described as a perspective projection
- Projective systems include perspectivities as a special case and **do** form a group
  – We now require 4 basis vectors (three axes plus an additional independent vector)
  – A model for linear transformations (also called collineations or homographies) on $P^n$ is GL(n+1) which is, of course, a group



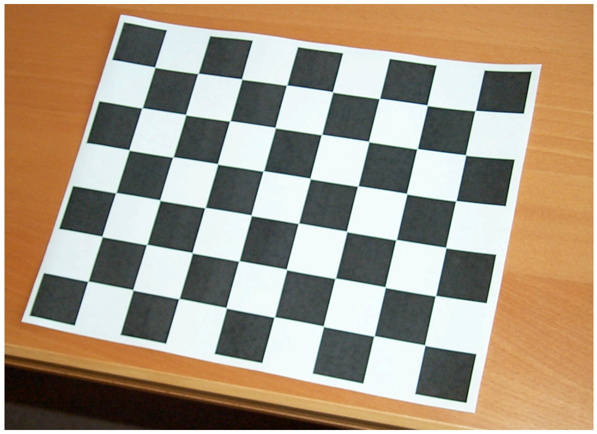
# Camera calibration

- Issues:
  – what are intrinsic parameters of the camera?
  – what is the camera matrix? (intrinsic+extrinsic)
- General strategy:
  – view calibration object
  – identify image points
  – obtain camera matrix by minimizing error
  – obtain intrinsic parameters from camera matrix
- Most modern systems employ the multi-plane method
  – avoids knowing absolute coordinates of calibration poitns
- Error minimization:
  – Linear least squares
    - easy problem numerically
    - solution can be rather bad
  – Minimize image distance
    - more difficult numerical problem
    - solution usually rather good, but can be hard to find
      – start with linear least squares
  – Numerical scaling is an issue

# Calibration – Problem Statement

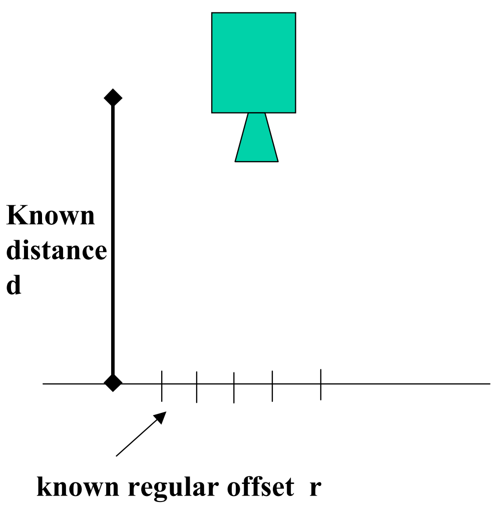
**The problem:**

Compute the camera intrinsic (4 or more) and extrinsic parameters (6) using only observed camera data.



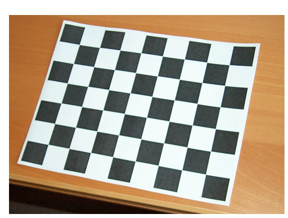


# CAMERA CALIBRATION: A WARMUP




$$\frac{rk_i}{d} = (x_i - o_x)s_x$$

$$\frac{r}{d} = (x_{i+1} - x_i)s_x$$



A simple way to get scale parameters; we can compute the optical center as the numerical center and therefore have the intrinsic parameters

# Calibration: Another Warmup

- Suppose we want to calibrate the affine camera and we know $u_i = A\ p_i + d$ for many pairs i
- m is mean of u's and q is mean of p's; note m = A q + d
- $U = [u_1 - m, u_2 - m, \ldots u_n - m]$ and $P = [p_1 - q, p_2 - q, \ldots p_n - q]$
- $U = A\ P \Diamond U\ P' (P\ P')^{-1} = A$
- d is now mean of $u_i - A\ p_i$



# Types of Calibration

- Photogrammetric Calibration
- Self Calibration
- Multi-Plane Calibration

# Photogrammetric Calibration

- Calibration is performed through imaging a pattern whose geometry in 3d is known with high precision.

- PRO: Calibration can be performed very efficiently
- CON: Expensive set-up apparatus is required; multiple orthogonal planes.

- Approach 1: Direct Parameter Calibration
- Approach 2: Projection Matrix Estimation

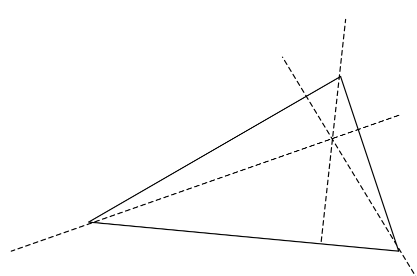


# The General Case

- Affine is “easy” because it is linear and unconstrained (note orthographic is harder because of constraints)

- Perspective case is also harder because it is both nonlinear and constrained

- Observation: optical center can be computed from the *orthocenter* of vanishing points of orthogonal sets of lines.



# Basic Equations

$$ {}^cT_w = (T_x, T_y, T_z)' $$

$$ {}^cR_w = (R_x, R_y, R_z)' $$

$$ {}^cp = {}^cR_w{}^wp +{}^c T_w $$

$$ u = -f\frac{R_xp + T_x}{R_zp + T_z} $$

$$ v = -f\frac{R_yp + T_y}{R_zp + T_z} $$



# Basic Equations

$$ u_{pix} = \frac{1}{s_x}u + o_x $$

$$ v_{pix} = \frac{1}{s_y}v + o_y $$

$$ \bar{u} = u_{pix} - o_x = -f_x\frac{R_xp + T_x}{R_zp + T_z} $$

$$ \bar{v} = v_{pix} - o_y = -f_y\frac{R_yp + T_y}{R_zp + T_z} $$

# Basic Equations

$$\bar{u}_i f_y(R_y p_i + T_y) = \bar{v}_i f_x(R_x p_i + T_x)$$
$$\bar{u}_i(R_y p_i - T_y) - \bar{v}_i \alpha(R_x p_i + T_x) = 0$$

$r = \alpha R_x$ and $w = \alpha T_x$

$t = R_y$ and $s = T_y$

one of these for each point

$A_i = (u_i p_i, u_i, -v_i p_i, -v_i)$ and $A[t, s, w, r]' = 0$



# Properties of SVD

- Recall the singular values of a matrix are related to its rank.
- Recall that Ax = 0 can have a nonzero x as solution only if A is singular.
- Finally, note that the matrix V of the SVD is an orthogonal basis for the domain of A; in particular the zero singular values are the basis vectors for the null space.
- Putting all this together, we see that A must have rank 7 (in this particular case) and thus x must be a vector in this subspace.
- Clearly, x is defined only up to scale.

# Basic Equations

$A_i = (u_i p_i, u_i, -v_i p_i, -v_i)$ and
$A[t, s, w, r]' = Am = 0$

Note that m is defined up a scale factor!

$A = UDV'$ and choose m as column of V corresponding to the smallest singular value



# Basic Equations

$A_i = (u_i p_i, u_i, -v_i p_i, -v_i)$ and
$A[t, s, w, r]' = Am = 0$

$||t|| = |\gamma|$ gives scale factor for solution
$||w|| = |\gamma|\alpha$

We now know $R_x$ and $R_y$ up to a sign and $\gamma$.
$R_z = R_x \times R_y$

We will probably use another SVD to orthogonalize this system (R = U D V'; set D to I and multiply).

# Last Details

- We still need to compute the correct sign.
  - note that the denominator of the original equations must be positive (points must be in front of the cameras)
  - Thus, the numerator and the projection must disagree in sign.
  - We know everything in numerator and we know the projection, hence we can determine the sign.

- We still need to compute $T_z$ and $f_x$
  - we can formulate this as a least squares problem on those two values using the first equation.

$$\bar{u} = -f_x \frac{R_x p + T_x}{R_z p + T_z} \rightarrow$$
$$\bar{u}(R_z p + T_z) = -f_x(R_x p + T_x)$$
$$f_x(R_x p + T_x) + \bar{u} T_z = -\bar{u} R_z p$$
$$A(f_x, T_z)' = b \rightarrow (f_x, T_z)' = (A'A)^{-1} A' b$$



# Direct Calibration: The Algorithm

1. Compute image center from orthocenter
2. Compute the A matrix (6.8)
3. Compute solution with SVD
4. Compute gamma and alpha
5. Compute R (and normalize)
6. Compute $f_x$ and and $T_z$
7. If necessary, solve a nonlinear regression to get distortion parameters

## Indirect Calibration: The Basic Idea

- We know that we can also just write
  - $\mathbf{u}_h = M\,\mathbf{p}_h$
  - x = (u/w) and y = (v/w), $\mathbf{u}_h = (u,v,1)'$
  - As before, we can multiply through (after plugging in for u,v, and w)
- Once again, we can write
  - A m = 0
- Once again, we use an SVD to compute m up to a scale factor.



## Getting The Camera Parameters

$$M = \begin{bmatrix} -f_x R_x + o_x R_z & -f_x T_x + o_x T_z \\ -f_y R_y + o_y R_z & -f_y T_y + o_y T_z \\ R_z & T_z \end{bmatrix}$$

We'll write

$$M = \begin{bmatrix} q_1 & \\ q_2 & q_4' \\ q_3 & \end{bmatrix}$$

# Getting The Camera Parameters

$$M = \begin{bmatrix} -f_x R_x + o_x R_z & -f_x T_x + o_x T_z \\ -f_y R_y + o_y R_z & -f_y T_y + o_y T_z \\ R_z & T_z \end{bmatrix}$$

We'll write

$$M = \begin{bmatrix} q_1 & \\ q_2 & q_4' \\ q_3 & \end{bmatrix}$$

FIRST:

$|q_3|$ is scale up to sign; divide by this value

$M_{3,4}$ is $T_z$ up to sign, but $T_z$ must be positive; if not divide M by -1

$o_x = q_1 \cdot q_3$
$o_y = q_2 \cdot q_3$
$f_x = (q_1 \cdot q_1 - o_x^2)^{1/2}$
$f_y = (q_2 \cdot q_2 - o_y^2)^{1/2}$

THEN:

$R_y = (q_2 - o_y R_z)/f_y$
$R_x = R_y \times R_z$
$T_x = -(q_{4,1} - o_x T_z)/f_x$
$T_y = -(q_{4,2} - o_y T_z)/f_y$

Finally, use SVD to orthogonalize the rotation,



# Self-Calibration

- Calculate the intrinsic parameters solely from point correspondences from multiple images.
- Static scene and intrinsics are assumed.
- No expensive apparatus.
- Highly flexible but not well-established.
- Projective Geometry – image of the absolute conic.

# Model Examples: Points on a Plane

- Normal vector n =($n_x$,$n_y$,$n_z$,0)'; point P = ($p_{\mathbf{x}}$,$p_y$,$p_z$,1)
  plane equation: n ¢ P = d
  – w/o loss of generality, assume $n_z \neq 0$
  – Thus, $p_z = a\ p_x + b\ p_y + c$; let B = (a, b, 0, c)
  – Define P' = ($p_x$,$p_y$,0,1)
  – P = P' + (0,0,B P',0)

- Affine: **u** = A P**,** A a 3 by 4 matrix
  – u = $A_{1,2,4}$ P' + $A_3$ B P' = $A_{3x3}$ $P_{3£1}$
  – Note that we can now *reproject* the points u and group the projections --- in short projection of projections stays within the affine group

- Projective **p** = M **P**, M a 4 by 3 matrix
  – p = $M_{1,2,4}$ P' + $M_3$ B P' = M $P_{3£1}$
  – Note that we can now *reproject* the points p and group the resulting matrices --- in short projections of projections stays within the projective group



# Multi-Plane Calibration

- Hybrid method: Photogrammetric and Self-Calibration.
- Uses a planar pattern imaged multiple times (inexpensive).
- Used widely in practice and there are many implementations.
- Based on a group of projective transformations called homographies.

- m be a 2d point [u v 1]' and M be a 3d point [x y z 1]'.

- Projection is

$$s\tilde{m} = A\begin{bmatrix} R & T \end{bmatrix}\tilde{M}$$

# Review: Projection Model

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} \Rightarrow \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}_{pix} = \begin{pmatrix} s_u & 0 & o_u \\ 0 & s_v & o_v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}_{mm} = Ap$$



# Review: Projection Model

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} = I\left[R,T\right] \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix}_{pix} = \begin{pmatrix} fs_u & 0 & o_u \\ 0 & fs_v & o_v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}_{mm} = \begin{pmatrix} \alpha & \gamma & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} Ap$$

# Result

- We know that $[h_1 \quad h_2 \quad h_3] = sA[r_1 \quad r_2 \quad t]$
- From one homography, how many constraints on the intrinsic parameters can we obtain?
  - Extrinsics have 6 degrees of freedom.
  - The homography supplies 8 values.
  - Thus, we should be able to obtain 2 constraints per homography.
- Use the constraints on the rotation matrix columns…



# Planar Homographies

- First Fundamental Theorem of Projective Geometry:
  - There exists a unique homography that performs a change of basis between two projective spaces of the same dimension.

$$s\tilde{m} = H\tilde{M}$$

  - Notice that the homography is defined up to scale (s).
- In P(2), we have
  - p' = H p for points p
  - u' = $H^t$ u for lines u
- Note to define the homography, we need three basis vectors *plus* the unit point!

# Planar Homographies

- First Fundamental Theorem of Projective Geometry:
  - There exists a unique homography that performs a change of basis between two projective spaces of the same dimension.

$$
\begin{aligned}
s[u \quad v \quad 1]^T &= A[r_1 \quad r_2 \quad r_3 \quad t][X \quad Y \quad Z \quad 1]^T \\
s[u \quad v \quad 1]^T &= A[r_1 \quad r_2 \quad r_3 \quad t][X \quad Y \quad 0 \quad 1]^T \\
s[u \quad v \quad 1]^T &= A[r_1 \quad r_2 \quad t][X \quad Y \quad 1]^T \\
s[u \quad v \quad 1]^T &= H[X \quad Y \quad 1]^T
\end{aligned}
$$

  - Projection Becomes

$$s\tilde{m} = H\tilde{M}$$

  - Notice that the homography is defined up to scale (s).



# Estimating A Homography

- Here is what looks like a reasonable recipe for computing homographies:
  - Planar pts $(x_1;y_1;1, x_2; y_2; 1, .... x_n;y_n;1) = X$
  - Corresponding pts $(u_1;v_1;1,u_2;v_2;1,...u_n;v_n;1) = U$
  - U = H X
  - $U X' (X X')^{-1} = H$

- The problem is that X will not be full rank (why?). So we'll have to work a little harder ...

# Computing Intrinsics

- Rotation Matrix is orthogonal….

$$r_i^T r_j = 0$$
$$r_i^T r_i = r_j^T r_j$$

- Write the homography in terms of its columns…

$$\begin{aligned} h_1 &= sAr_1 \\ h_2 &= sAr_2 \\ h_3 &= sAt \end{aligned}$$



# Computing Intrinsics

- Derive the two constraints:

$$\begin{aligned} h_1 &= sAr_1 \\ \frac{1}{s}A^{-1}h_1 &= r_1 \\ \frac{1}{s}A^{-1}h_2 &= r_2 \end{aligned}$$

$$\begin{aligned} r_1^T r_2 &= 0 \\ h_1^T A^{-T} A^{-1} h_2 &= 0 \end{aligned}$$

$$\begin{aligned} r_1^T r_1 &= r_2^T r_2 \\ h_1^T A^{-T} A^{-1} h_1 &= h_2^T A^{-T} A^{-1} h_2 \end{aligned}$$

# Closed-Form Solution

Let $B = A^{-T}A^{-1} = \begin{bmatrix} \frac{1}{\alpha^2} & -\frac{\gamma}{\alpha^2\beta} & \frac{v_0\gamma - u_0\beta}{\alpha^2\beta} \\ -\frac{\gamma}{\alpha^2\beta} & \frac{\gamma^2}{\alpha^2\beta^2} + \frac{1}{\beta^2} & -\frac{\gamma(v_0\gamma - u_0\beta))}{\alpha^2\beta^2} - \frac{v_0}{\beta^2} \\ \frac{v_0\gamma - u_0\beta}{\alpha^2\beta} & -\frac{\gamma(v_0\gamma - u_0\beta))}{\alpha^2\beta^2} - \frac{v_0}{\beta^2} & \frac{(v_0\gamma - u_0\beta)^2}{\alpha^2\beta^2} + \frac{v_0^2}{\beta^2} + 1 \end{bmatrix}$

- Notice B is symmetric, 6 parameters can be written as a vector b.
- From the two constraints, we have $h_i^T B h_j = v_{ij}^T$

$$\begin{bmatrix} v_{ij}^T \\ (v_{11} - v_{22})^T \end{bmatrix} b = 0;$$

- Stack up n of these for n images and build a 2n*6 system.
- Solve with SVD (yet again).
- Extrinsics “fall-out” of the result easily.



# Non-linear Refinement

- Closed-form solution minimized algebraic distance.
- Since full-perspective is a non-linear model
  - Can include distortion parameters (radial, tangential)
  - Use maximum likelihood inference for our estimated parameters.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} ||m_{ij} - \hat{m}(A, R_k, T_k, M_j)||^2$$

# Multi-Plane Approach In Action

- …if we can get matlab to work…

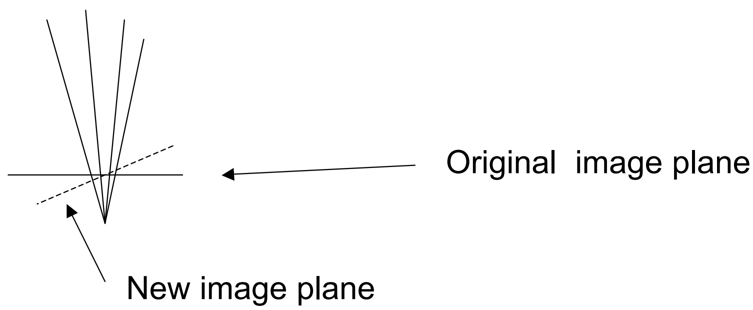


# Calibration Summary

- Two groups of parameters:
  - internal (intrinsic) and external (extrinsic)
- Many methods
  - direct and indirect, flexible/robust
- The form of the equations that arise here and the way they are solved is common in vision:
  - bilinear forms
  - Ax = 0
  - Orthogonality constraints in rotations
- Most modern systems use the method of multiple planes (matlab demo)
  - more difficult optimization over a large # of parameters
  - more convenient for the user

# An Example Using Homographies

- Image rectification is the computation of an image as seen by a rotated camera
  - The computation of the planar reprojection is a homography
  - we'll show later that depth doesn't matter when rotating; for now we'll just use intuition






# Rectification Using Homographies

- Pick a rotation matrix R from old to new image

- Consider all points in the image *you want to compute*; then
  - construct pixel coordinates x = (u,v,1)
  - K maps unit focal length metric coordinates to pixel (normalized camera)
  - $x' = K\,R^t K^{-1}\,x$ ◊ $x' = H\,x$

pixel coordinates

pixel coordinates

rotation

pixel to euclidean coords

euclidean to pixel coords

- Sample a point x' in the original image for each point x in the new.

# Bilinear Interpolation

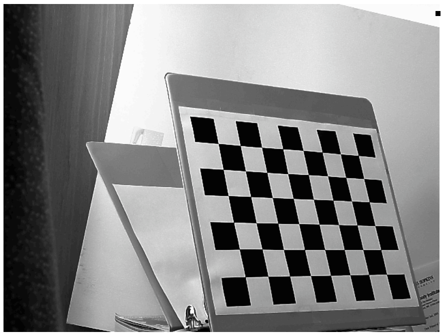
- A minor detail --- new value x' = (u',v',1) may not be integer
- let $u' = i + f_u$ and $v' = j+f_v$
- New image value $b = (1-f_u)((1-f_v)I(j,i) + f_v\, I(j+1,i)) + f_u((1-f_v)I(j,i+1) + f_v\, I(j+1,i+1))$



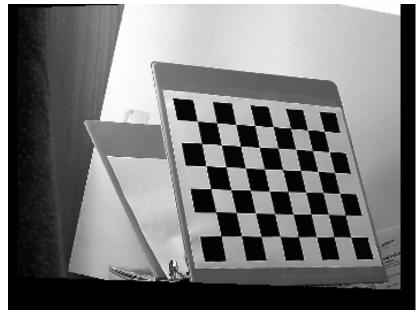
# Rectification: Basic Algorithm

- 1. Create a mesh of pixel coordinates for the rectified image
- 2. Turn the mesh into a list of homogeneous points
- 3. Project *backwards* through the intrinsic parameters to get unit focal length values
- 4. Rotate these values back to the current camera coordinate system.
- 5. Project them *forward* through the intrinsic parameters to get pixel coordinates again.
  - Note equivalently this is the homography $K R^t K^{-1}$ where K is the intrinsic parameter matrix
- 6. Sample at these points to populate the rectified image
  - typically use bilinear interpolation in the sampling

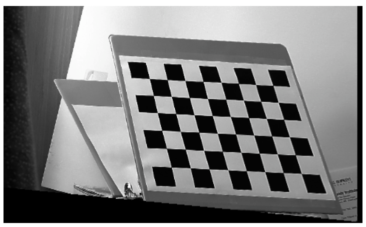
# Rectification Results

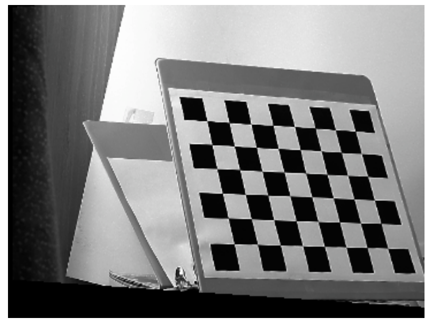


.2 rad



.4 rad





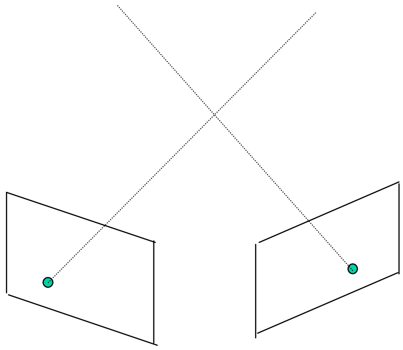
.6 rad

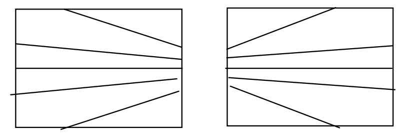


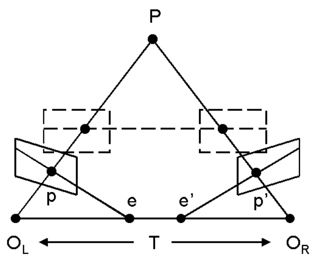
# “Homework” Problems

- Derive the relationship between the Plucker coordinates of a line in space and its projection in Plucker coordinates
- Show that the projection of parallel lines meet at a point (and show how to solve for the point)
- Given two sets of points that define two projective bases, show how to solve for the homography that relates them.
- Describe a simple algorithm for calibrating an affine camera given known ground truth points and their observation --- how many points do you need?

# Two-Camera Geometry

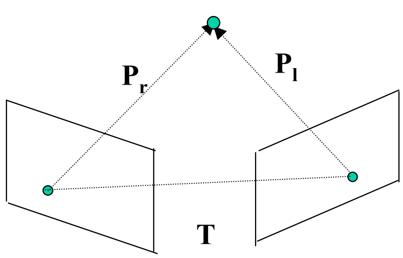









# E matrix derivation




$$P_r = R(P_l - T)$$

$$(P_l - T) \cdot (T \times P_l) = 0$$
$$P_r^t R (T \times P_l) = 0$$
$$P_r^t E P_l = 0$$

**where E = R sk(T)**

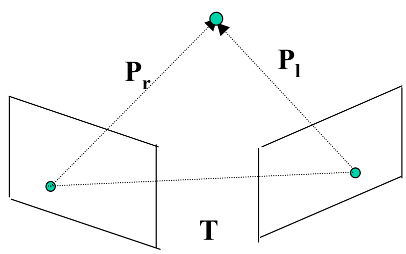
$$sk(T) = \begin{matrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{matrix}$$

**The matrix E is called the *essential* matrix and completely describes the epipolar geometry of the stereo pair**

# Fundamental Matrix Derivation

Note that E is invariant to the scale of the points, therefore we also have

$$p_r^t \, E \, p_l = 0$$




$$P_r = R(P_l - T)$$

where p denotes the (metric) image projection of P

Now if K denotes the internal calibration, converting from metric to pixel coordinates, we have further that

$$r_r^t \, K^{-t} \, E \, K^{-1} \, r_l = r_r^t \, F \, r_l = 0$$

where r denotes the *pixel* coordinates of p. F is called the *fundamental matrix*