Basic Electricity and Magnetism 3910

Current Flow in Ohmic Resistors

The general problem

Most materials are characterized by a bulk parameter called *resistivity*, symbolized by ρ. The resistivity can be thought of as a relationship between fields. If $\vec{\mathbf{J}}$ is the current density at some point in the material and $\vec{\mathbf{E}}$ is the electric field vector at that point, for many materials the two are related by "Ohm's Law,"

$$\vec{\mathbf{J}} = \vec{\mathbf{E}}/\rho \ . \tag{1}$$

The resistivity ρ is an intrinsic property of materials that indicates how easily current can be pushed through the material by an electric field. In common engineering units, ρ is measured in ohm·m or ohm·cm.

Ohm's law is not a fundamental law, like any of Maxwell's laws, but rather is a description of the behavior of certain materials called "ohmic materials." It is analogous to the way in which Hooke's law for springs gives an approximate description of the way in which certain materials (elastic materials) respond to applied stresses. The theoretical picture underlying Eq. (1) is based on the idea that current flow in ohmic materials is a diffusive motion of electrons driven by electric fields inside the materials. There are nonohmic materials in which Eq. (1) does not apply because $\vec{\mathbf{J}}$ has a very nonlinear dependence on $\vec{\mathbf{E}}$. In air, for instance, the current flow is negligible until $\vec{\mathbf{E}}$ builds up to a strength of $\sim 10^8$ volts/m. When this happens a spark jumps through the air, and $\vec{\mathbf{J}}$ is momentarily large. There are other materials ("anisotropic materials"), such as graphite, in which $|\vec{\mathbf{J}}|$ is proportional to $|\vec{\mathbf{E}}|$ but in which $\vec{\mathbf{J}}$ is not generally in the same direction as $\vec{\mathbf{E}}$, due to direction dependent properties of the material.

Figure 1 shows a rectilinear solid made of a uniform ohmic material. Let us suppose that the two ends of this solid have plates, called electrodes, that can be approximated as perfectly conducting. A resistive material with electrodes connected to it is called a resistor. The ends of the resistor must be electrical equipotentials since they are perfectly conducting electrodes. If a voltage difference ΔV is applied to the two electrodes, say be means of the attached wires shown, the electrical field $\vec{\mathbf{E}}$ will then be uniform inside the solid and hence the

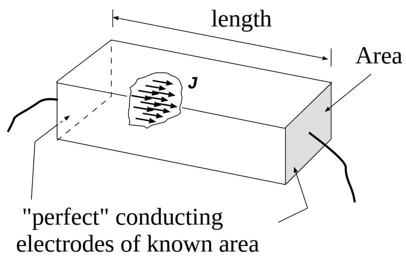


Figure 1: Uniform current flow in a constant cross section resistor.

current density $\vec{\mathbf{J}}$ will be uniform. The total current flow I through the resistor will be the magnitude of $\vec{\mathbf{J}}$ multiplied by the area of the ends. Thus we have

$$I = J \times \text{Area} = E \times \text{Area}/\rho \ .$$

But the voltage difference ΔV will simply be E times the length of the resistor so that $E = \Delta V/\text{length}$. The resulting relationship of current and voltage difference is usually written as

$$I = \Delta V/R\,, \tag{2}$$

where R, the "resistance" of the resistor is

$$R = \rho \times \text{length}/\text{Area}. \tag{3}$$

Note that Eq. (2) which, like Eq. (1) is also called Ohm's law, states that for a given resistor the current through the resistor is proportional to the voltage across it.

In circuits, elements that are used for the specific purpose of adding resistance are often cylindrical (carbon resistors). In circuit diagrams, resistors are usually pictured as sawtoothed circuit elements, like the resistors with resistance R_1 and R_2 shown in Fig. 2. The two basic connections are also shown in that figure. For a series connection the total equivalent resistance is $R_1 + R_1$ and is greater than the resistance of either resistor; for a parallel connection the total resistance is $R_1R_2/(R_1+R_1)$ and is smaller than the resistance of either resistor. We assume that you know the rules for finding the resistance of combinations of resistors, and that you know that these rules are based on two simple ideas: (i) In a series combinations, the total voltage drop is the sum of the voltage drop across each resistor, and the same current flows through both resistors. (ii) In a parallel combination, the voltage drop across both resistors is the same, and the total current is the sum of the individual currents through each resistor.

The idea of a resistor circuit element is based on an assumption. It is assumed that the resistor is surrounded by material (air, say) that effectively has

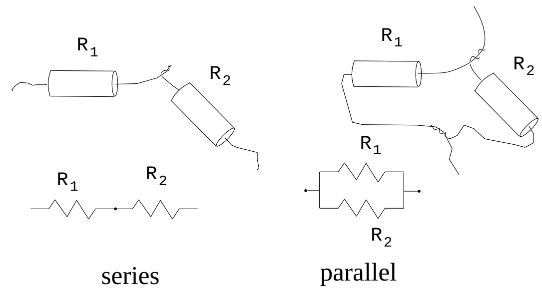


Figure 2: Series and parallel combinations of resistor circuit elements.

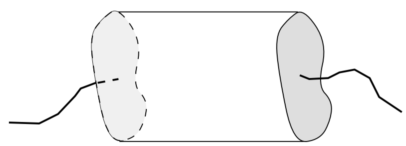


Figure 3: A resistor with a constant cross sectional area.

an infinite resistance, so that current flows, from electrode to electrode, only through the resistor. For Eq. (2) to be correct, i.e. , for it to give the correct value for the proportionality constant in Ohm's law, something else is needed: The current flow must be uniform. It should be intuitively clear that this will be the case if the resistor has constant cross section and has perfectly conducting electrodes parallel to a cross section. The resistor need not be rectilinear, but may look, e.g. , like the object in Fig. 3.

What happens if the cross section is not constant? Figure 4 shows such a resistor, a truncated cone made of an ohmic material. We assume that the flat end faces of the cone are perfectly conducting electrodes. In many elementary textbooks students are told that the problem can be solved by imagining the cone to be sliced into infinitesimal disks with flat faces. The infinitesimal resistance for such a disk is given by Eq. (3). The contributions from each infinitesimal disk are then added (integrated) to get the total resistance.

This turns out to be incorrect; it gives too low a value for the resistance. The flow of current through the cone is not the same as current flow through the disks. Consider the disk faces. In the slice-and-add model, current flows perpendicular to the disk faces. This would correspond to current flowing parallel to the axis of the cone. But it is impossible for current to flow parallel to the axis everywhere inside the cone.

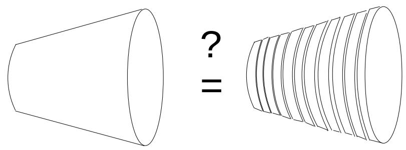


Figure 4: A resistor with a varying cross sectional area.

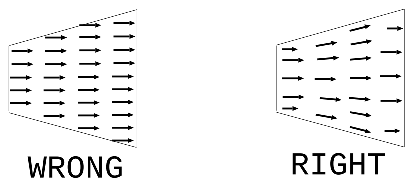


Figure 5: Incorrect (left) and correct (right) current flows in a resistor with the shape of a truncated cone.

If the current everywhere *were* flowing parallel to the axis, as shown on the left hand side of Fig. 5, the current would be flowing into the resistor through the curved side of the cone! It is clear that $\vec{\mathbf{J}}$ must be parallel to the side of the resistor. Since the $\vec{\mathbf{E}}$ is in the same direction as $\vec{\mathbf{J}}$, it follows that $\vec{\mathbf{E}}$ must also be parallel to the side of the resistor. Of course, there is another requirement that these fields must satisfy. Since the flat faces of the truncated cone, the electrodes, are equipotentials, the $\vec{\mathbf{E}}$ field, and hence the $\vec{\mathbf{J}}$ field, must be perpendicular to the electrodes. The right hand side of Fig. 5 shows the correct pattern of $\vec{\mathbf{J}}$ and/or $\vec{\mathbf{E}}$. Notice that the two conditions are incompatible at the edge of the electrodes, where the sides meet the electrodes at an obtuse anlge. As a result the fields must be zero at the edge.

We have the two surface requirements for the fields, but we need to know what additional physics determines the fields inside. This turns out to follow from the fact that the divergence of $\vec{\mathbf{J}}$ gives the rate at which charge density is decreasing. If we are working with a steady state problem, charge density cannot change, and hence the divergence of $\vec{\mathbf{J}}$ must vanish, or equivalently (since $\vec{\mathbf{J}}$ and $\vec{\mathbf{E}}$ are related by a consant),

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = 0 \ . \tag{4}$$

Since we can relate $\vec{\mathbf{E}}$ to the electrostatic potential by $\vec{\mathbf{E}} = -\vec{\nabla}\Phi$, and therefore Eq. (4) is equivalent to

$$\nabla^2 \Phi = 0 \ . \tag{5}$$

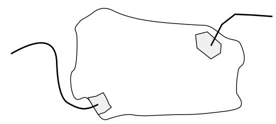


Figure 6: A true 3D problem in computiting resistance.

One more piece of information must be given to the mathematics: The difference in the electrostatic potential Φ from one electrode to the other is the voltage applied across the resistor.

This completes the mathematical description of the problem. We summarize it here:

1. The Laplace equation $\nabla^2\Phi = 0$ must be solved for the electrostatic potential Φ.

2. The values of Φ at the electrodes must correspond to the voltages applied to the electrodes.

3. The direction of $\vec{\nabla}\mathbf{\Phi}$ at the sides must be parallel to the sides.

It appears that we have left out the requirement that the direction of $\vec{\nabla}\mathbf{\Phi}$ at the electrodes must be perpendicular to the electrodes. But this is built into requirement 2 above; if the electrodes are equipotentials, i.e., surfaces of constant Φ, then $\vec{\nabla}\mathbf{\Phi}$ is automatically perpendicular to the electrodes.

It is intuitively obvious that for the constant cross section resistors of Figs. 1 and 3 the solution to the problem is that corresponding to $\vec{\nabla}\mathbf{\Phi}$ being a constant vector (constant magnitude, constant direction) inside the resistor. For resistors without constant cross section, like the truncated cone in Fig. 4, our mathematical problem for Φ must in general be solved with numerical methods on a computer. Fortunately, computational methods of solution are highly developed. Equations like Laplace's equation are of great importance in many technological applications, and much effort has gone into the development of numerical methods and convenient interfaces. The package "*QuickField*" can solve problems such as that of the truncated cone. The cone problem, however, has the simplifying feature that it has an axis of symmetry. Though the cone itself is 3 dimensional, the problem of analyzing fields inside it is 2D (two dimensional). In cylindrical coordinates $\{r, \varphi, z\}$, for example, the potential Φ would depend only on r and z; finding it would be a 2D problem. Such 2D problems, in the early 21$^{\text{st}}$ century, can be solved with reasonable accuracy even on medium size PCs. By contrast, a truly 3D, like that in Fig. 6 requires a powerful workstation for a solution with reasonable accuracy.

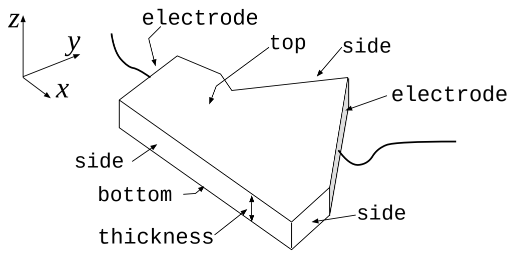


Figure 7: A "flat" resistor.

Two dimensional resistors

We will concentrate here on a class of 2D resistors here, a class we will call "flat" resistors. Figure 7 shows an example. These resistors have parallel faces that we (somewhat arbitrarily) shall call the top and bottom, and the thickness of the resistors, between the top and bottom faces, is constant. (In Fig. 7 the top and bottom faces are parallel to the xy plane.) The "sides" of the resistors are perpendicular to the top and bottom, but may have any relationship to other sides. Two of the sides have perfect conducting electrodes attached.

One of the reasons for our interest in flat resistors is that they present us with 2D problems. The current flow has no z component, in our figure. It follows that Φ will be a function only of x and y. A second reason we are interested, is that such resistors are easy to fabricate. By depositing ohmic materials as a layer, we can make flat resistors with shapes of our choice, and get an opportunity to compare measurement and computation. There is a very important third reason: Flat resistors are very common in technological applications, this is especially true if the thickness of these resistors is small. Such thin layer resistors can be used, for instance, as a model for the conducting traces on a printed circuit board. Although such traces are made of a material of low resistivity (generally copper) the traces are very thin, and hence their resistance is not negligible.

Usually when we deal with flat resistors, the thickness is fixed, and we characterize the resistance not with the bulk resistivity ρ, but with ρ divided by the thickness. We will call this quantity the "surface resistivity" $\rho_{\rm surf}$ of the material. Since ρ has units of ohms$\times$length, $\rho_{\rm surf}$ has units of ohms.

The use of $\rho_{\rm surf}$ allows us to focus attention on the shape of the top (or identical bottom) of a thin layer resistor. For a rectangular shape, like that in Fig. 8, the resistance is simply

$$R = \rho_{\rm surf} L/W \ . \tag{6}$$

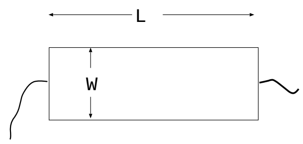
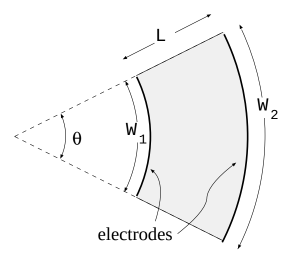


Figure 8: A rectangular 2D resistor.

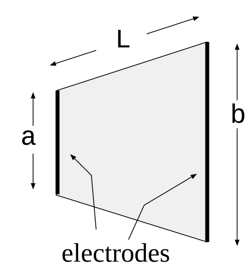
This is, in fact, just a special case of Eq. (3). For a square we have that $R = \rho_{\text{surf}}$. The unit of surface resistivity is, in fact, sometimes called not "ohms" but "ohms per square" to help distinguish this intrinsic property of surfaces from the resistance, measured in ohms, of a particular 2D resistor.

For shapes other than rectangular, closed-form solutions in general do not exist. The mathematical problem of solving is precisely that of the three steps given following Eq. (5). But, as explained above, the problems are now 2D. The electric field does not vary from top to bottom.

Problems

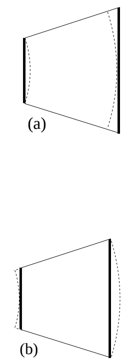


Problem 1 A 2D resistor has a shape bounded by two circular arcs, of arc length W_1 and W_1. The radial distance between the two arcs is L. The electrode edges are the circular arcs. It should be clear that the exact resistance between the two electrodes is of the form $R = \rho_{\text{surf}} L/f(W_1, W_2)$, where $f(W_1, W_2)$ is some function of W_1 and W_2. Find $f(W_1, W_2)$ and show explicitly that $f \to W_1$ in the limit that the fractional difference between W_1 and W_2 is small.



Problem 2 The trapezoidal 2D resistor shown in the figure has surface resistivity ρ_{surf} and dimensions shown. The resistor is symmetric; it looks the same if flipped about a horizontal axis. The resistance cannot be found with an elementary method. One way of getting an approximation is to slice the trapezoid into differential slices (analogous to the slicing of the cone in Fig. 4). Calculate the resistance of the slices added in series and give the result as a function of ρ_{surf}, L, a, and b. Is this approximation an overestimate or an underestimate of the true resistance?

PROBLEM 3 In Problem 1 an exact solution was found for a circular 2D resistor. A resistor of this type can be inscribed inside the trapezoidal resistor of Problem 2, as shown in part (a) of the figure. A resistor of this type can can also be used to surround the trapezoidal resistor of Problem 2, as shown in part (b) of the figure. Use the resistance of these circular resistors as approximations to the resistance of the trapezoidal resistor. Explain how this gives an upper bound and a lower bound on the resistance. In the particular case $2a = b = L$ compare these approximations with the "slice and add" approximation of Problem 2.



Computing 2D Resistance

As previously explained, the problem of computing resistance is approached by specifying the voltage on the two electrodes and then solving the following mathematical problem:

1. The Laplace equation $\nabla^2\Phi = 0$ must be solved for the electrostatic potential Φ.

2. The values of Φ at the electrodes must correspond to the voltages applied to the electrodes.

3. The direction of $\vec{\nabla}\mathbf{\Phi}$ at the sides must be parallel to the sides.

Once the potential Φ is found, the electric field $\vec{\mathbf{E}} = -\vec{\nabla}\Phi$ is straightforward to calculate, and the current density $\vec{\mathbf{J}} = \vec{\mathbf{E}}/\rho$ follows immediately.

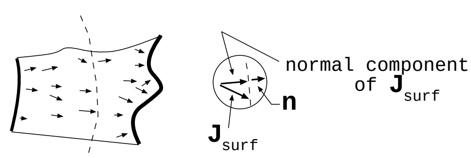


Figure 9: Finding the current in a 2D flow.

In the case of 2D current flow it is useful to introduce a surface density of current $\vec{\mathbf{J}}_{\rm surf}$, defined to be the current density multiplied by the (uniform) thickness of the layer through which the current passes. Since the units of current density are current per unit area (amps/m^2, or amps/cm^2), the units of $\vec{\mathbf{J}}_{\rm surf}$ are current per unit length (amps/m, or amps/cm). The surface current density is related to the electric field by

$$\vec{\mathbf{J}}_{\rm surf} = \vec{\mathbf{E}}/\rho \times \text{thickness} = \vec{\mathbf{E}}/\rho_{\rm surf} \ . \tag{7}$$

The total current flowing through the resistor is the current passing any surface through the resistor. For a 2D resistor this can be thought of as the surface current through any curve across the resistor. The dashed line in Fig. 9 is such a curve. The unit vector $\vec{\mathbf{n}}$ that is normal to the curve is shown in the detailed close up. The component of $\vec{\mathbf{J}}_{\rm surf}$ parallel to the curve does not carry current across the curve. Only the normal component $\vec{\mathbf{J}}_{\rm surf} \cdot \vec{\mathbf{n}}$ does, and this component therefore tells us what the current, per unit curve length is at that point on the curve. To get the total current passing through the curve we add all the contributions from each element, of length $d\ell$ of the curve, the summation of these contributions is the integral

$$I = \int \vec{\mathbf{J}}_{\rm surf} \cdot \vec{\mathbf{n}} \; d\ell \; . \tag{8}$$

The integral extends over that portion of the curve that is inside the resistor. (Since $\vec{\mathbf{J}}_{\rm surf} = 0$ outside the resistor, the integral can just as well be taken to extend over the entire curve.)

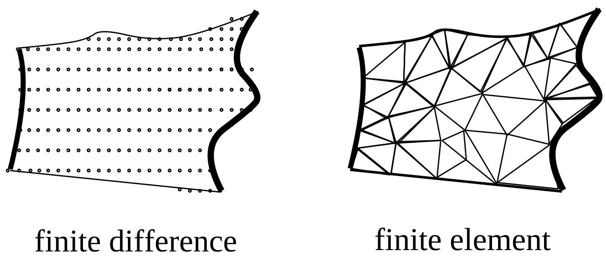


Figure 10: Computational schemes

There are two basic classes of schemes for numerically computing a solution to Laplaces equation (and other partial differential equations). In the "finite difference" approach, a grid of points is defined in the region to be solved, as pictured on the left in Fig. 10. Rather than try to compute the function $\Phi(x, y)$, we try to find an approximation to the values of Φ only on these grid points. The differential equation is replaced by an approximation involving the difference of Φ values on nearby points, and the differences Δx, and Δy, separating the grid points. These finite (rather than infinitesimal) differences give the finite difference method its name. The method leads to a very large set of linear equations for the large set of unknowns.

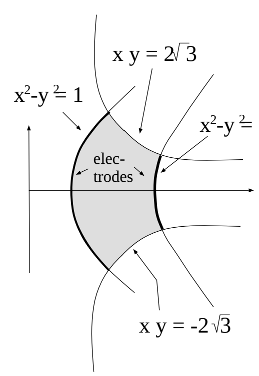
In the finite difference method the region to be solved is divided into a large number of small polygons. In the illustration on the right side of Fig. 10 the interior of the resistor is divided into triangles. A simple form of the solution inside each polygon is assumed, often just a linear form $a + bx + cy$, with undertermined coefficients in each polygon. The coefficients are then adjusted to satisfy approximations to the boundary conditions of the problem, to have the functions continuous and smooth at the edges of the polygons, and to give the best (in some sense) approximation to a solution of the original problem. The equations for finding the coefficients turn out to be linear, but they are a very large set of equations, since there are many unknown coefficients.

Finite difference methods are much easier to program for a specific problem. Finite element methods usually use packaged software. The big advantage of finite element methods is the ease of building in boundary conditions. In both finite difference and finite element methods there is a need to solve a very large set of linear equations, typically much too large for a straightforward method. Much of the recent progress in solving partial differential equations has involved improvements in handling this set of equations. There are, however, other methods besides finite difference and finite element that can be very powerful in the right applications. You may come across the names “multigrid methods” and “spectral methods” in connection with numerical solutions of partial differential equations that arise in technological applications.

In this course we will be using the software package *QuickField* to solve a range of 2D problems involving the solution of partial differential equations. *QuickField* uses a finite element method and building a mesh of polygons will be an obvious step in your solution. One of the options in using *QuickField* to solve a problem is that of specifying the density of mesh polygons, rather than having *QuickField* do it automatically. As you probably would guess, a larger number of polygons results in a greater accuracy of solution, but a longer time for solution. *QuickField* not only solves the problem for you but provides a GUI with a wide range of tools for analyzing the solution. In setting up a 2D current flow problem you’ll need to specify the voltage on the electrodes and to specify that there is no current flowing across the sides of the 2D resistor. In analyzing the solution you’ll use the `contour` menu to construct a curve through the current flow like that in Fig. 9. The `integral values` option will then tell you the current flowing across your contour. From the voltage difference you have specified, and the current flowing you can immediately infer the resistance of your 2D resistor model. One interesting feature of current flow problems in *QuickField* is the option to choose anisotropic resistance, i.e. , different surface resistivity in different directions. This feature will be exploited in the problems below.

The PHYCS 3910 notes “A Quick Guide to QuickField” give you more details on the procedures to follow in using *QuickField*.

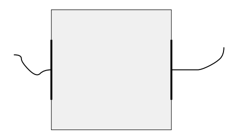
Problems



Problem 1 A 2D resistor has a surface resistivity ρ_{surf} =100 ohms. It has electrodes, as shown at the hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 1$, where all length units are cm. The sides of the resistor are the hyperbolas $xy = \pm 2\sqrt{3}$. (i) Give a rough estimate of the resistance between the electrodes. (ii) Suppose the left electrode is at +100 volt and the right electrode is at +900 volts. Find an expression for $\vec{\mathbf{E}}$ at any x, y position inside the 2D resistor. (Hint: $\Phi = k(x^2 - y^2)$ solves Laplace’s equation.) (iii) Sketch (reasonably accurately) the equipoltentials in the resistor, and the lines of current flow. (A line of current flow is a curve to which $\vec{\mathbf{J}}_{\text{surf}}$ is tangent.) (iv) Find $\vec{\mathbf{J}}_{\text{surf}}$ at every point along the line $x = 3$. (v) Find the total current through the line $x = 3$ and infer the exact resistance of the 2D resistor.

Problem 2 The problem of a trapezoidal resistor of length L and sides of length a and b was introduced in Problem 2 of the previous section. For dimensions $2a = b = L$, and surface resistivity ρ_{surf} =100 ohms find the resistance as accurately as possible using *QuickField*. Compare your answer to the estimates derived in Problems 2 and 3 of the previous sections. Is your *QuickField* answer between the upper and lower bounds you found?

Problem 3 A 2D resistor has electrodes attached to opposing faces of a square of a thin resistive layer. If the electrodes were the same length as the edges the resistance would be $R = \rho_{\text{surf}}$, but the electrodes in this problem cover only the middle half of the opposing edges. The resistance can be written as $R = k\rho_{\text{surf}}$, where k is some number. (i) Will k be larger or smaller than unity? (ii) Sketch the current flow in the resistor. (iii) Make a rough estimate of the resistance. (iv) Using *QuickField* compute the resistance.



Problem 4 Figure 4 illustrates the "slice-and-add" method of (incorrectly) finding the resistance of a truncated cone. (i) Apply this method to the problem of a cone for which the length L (i.e., the distance between electrodes) is the same as the radius of the larger electrode, and twice the radius of the smaller electrode. Express the answer in terms of ρ and L. (ii)Use the axially symmetric capability of *QuickField* to compute the resistance of the same truncated cone, and compare your answer with that of part (i).

Laboratory/Modelling Problem You will be given a glass slide containing a trapezoidal 2D resistor, and a rectangular resistor. (i) Find the surface resistivity ρ_{surf} by making measurements on the rectangular resistor. (ii) Measure the resistance of the trapezoidal resistor. (iii) Make measurements of the dimensions of your trapezoidal resistor. (iv) Using the dimensions you have measured find the resistance of the trapezoidal resistor by modeling it with *QuickField*. Write a comparison of your measured value and your modeled value, including estimates such as those of the previous problem. (v) For the same *QuickField* model display the intensity of the current density. Print out a picture of the display and add it to your report. (v) Put a thin coat of temperature sensitive LCD paint over your trapezoidal resistor. When it has dried, adjust a current through it so that you get as wide a range of colors as possible. Using the digital camera take a picture of the temperature profile and print out the result. Add the printout to your report and give a discussion of the comparison of the current density picture and the temperature profile.