PROCESS DYNAMICS: LAPLACE TRANSFORMS

M.T. Tham
Dept. of Chemical and Process Engineering
University of Newcastle upon Tyne

INTRODUCTION

It is always useful, and often essential, to analyse the performance capabilities and the stability of a proposed system before it is build or implemented. Many analysis techniques centre around the use of transformed variables to facilitate mathematical treatment of the problem. In the analysis of continuous time dynamical systems, the use of ***Laplace Transforms*** predominates.

Applying Laplace Transforms is analogous to using logarithms to simplify certain types of mathematical manipulations and solutions. By taking logarithms, numbers are transformed into powers of 10 or some other base, e.g. natural logarithms. As a result of the transformations, mathematical multiplications and divisions are replaced by additions and subtractions respectively. Similarly, the application of Laplace Transforms to the analysis of systems which can be described by linear, ordinary time differential equations overcomes some of the complexities encountered in the time-domain solution of such equations.

Laplace Transforms are used to convert time domain relationships to a set of equations expressed in terms of the Laplace operator '*s*'. Thereafter, the solution of the original problem is effected by simple algebraic manipulations in the 's' or Laplace domain rather than the time domain. The Laplace Transform of a time variable $f(t)$ is defined as:

$$F(s) = L\{f(t)\} = \int_0^t f(t)e^{-st}dt$$

where $L\{.\}$ is used to denoted the transformation.

Basic Properties of the Laplace Transform

The following are some of the fundamental properties of Laplace Transforms:

P1) The Laplace Transformation is linear, i.e.

$$L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\} = F_1(s) + F_2(s)$$

and $L\{kf(t)\} = kL\{f(t)\} = kF(s) \quad k = \text{constant}$

P2) Laplace Transformations of derivatives are given by the following:

$$L\{df(t)/dt\} = L\{f'(t)\} = sF(s) - f(0)$$

where $f(0)$ is the initial value of $f(t)$, at $t = 0$.

$$L\{d^2 f(t)/dt^2\} = L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

In general,

$$L\{d^n f(t)/dt^n\} = L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - \ldots - f^{n-1}(0)$$

P3) Laplace Transforms of integrals are given by:

$$L\{f^{-1}(t)\} = \lfloor F(s) - f^{-1}(0) \rfloor / s$$

In general,

$$L\{f^{-n}(t)\} = F(s)/s^n + f^{-1}(0)/s^n + f^{-2}(0)/s^{n-1} + \ldots + f^{-n}(0)/s$$

P4) The 'Final Value' theorem states that:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

and facilitates the determination of the value of $f(t)$ as time tend towards infinity, i.e. the steady-state value of $f(t)$.

P5) The 'Initial Value' theorem states that:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

and allows the determination of the value of $f(t)$ at time $t = 0^+$, i.e. at a time instant immediately after time $t = 0$.

Properties P1 to P4 are the most often used in systems analysis.

To return to the time-domain from the Laplace domain, ***inverse Laplace Transforms*** are used. Again this is analogous to the application of anti-logarithms and as in the use of logarithms, tables of Laplace Transform pairs help to simplify the task (see Table 1). More comprehensive lists of Laplace Transform pairs may be found in standard Control Engineering texts (e.g. Di'Stephano and colleagues, 1967; Kuo, 1980).

	Functions of time, *f(t)*	**Laplace Transforms of *f(t)*, *L*{*f(t)*}**
1.	$f(t)$	$F(s)$
2.	$x(t)+y(t)$	$X(s)+Y(s)$
3.	$k.f(t)$	$k.F(s)$
4.	$df(t)/dt$	$sF(s)-f(0)$
5.	$d^n f(t)/dt^n$	$s^n F(s)-s^{n-1}f(0)-s^{n-2}f^1(0)-\ldots-f^{n-1}(0)$
6.	$\int_0^t f(t).dt$	$F(s)/s$
7.	1	$1/s$
8.	t	$1/s^2$
9.	e^{-at}	$\frac{1}{(s+a)}$
10.	te^{-at}	$\frac{1}{(s+a)^2}$
11.	$1-e^{-at}$	$\frac{a}{s(s+a)}$
12.	$f(t-a), \quad t>a$	$e^{-as}F(s)$

Table 1. Table of Laplace Transformations

Illustrative Example

The following simple example illustrates the use of Laplace Transforms in systems analysis, showing how it is used to solve a linear ODE. Consider the first-order process:

$$\tau\frac{dY(t)}{dt}+Y(t)=KU(t) \qquad (1)$$

Here, $Y(t)$ is the output variable while $U(t)$ is the 'forcing' input. The time-domain solution of this ODE is required and is to be found using Laplace Transforms. The first step is to convert the 'full-valued' variables $Y(t)$ and $U(t)$ to their respective 'deviation varibales', $y(t)$ and $u(t)$ via:

$$y(t) = Y((t) - Y_{ss} \qquad u(t) = U((t) - U_{ss}$$

where Y_{ss} is the steady-state value that $Y(t)$ will attain given a steady input U_{ss}. Further, it is usual to assume that the steady-state values are equal to the initial values of the respective variables. Therefore,

$$y(0) = u(0) = 0$$

Since the time derivatives of steady values are zero, the deviation variables can be substituted into Eq.(1) to yield:

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t) \qquad (2)$$

Application of properties P1 and P2 to Eq.(2) yields:

$$\tau sY(s) - y(0) + Y(s) = KU(s) \qquad (3)$$

The reason for the use of deviation variables now becomes clear. Since $y(0)$ is zero, Eq.(3) simplifies to:

$$\tau sY(s) + Y(s) = KU(s) \qquad (4)$$

Therefore, using deviation variables allow us to simplify Laplace Transforms because all initial conditions are zero.

The effect of the input on output is then arrived at by simple rearrangement of terms to yield:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{1 + \tau s} \qquad (5)$$

$G(s)$ is called the ***transfer function*** of the process under consideration and describes the relationship between the input $U(s)$ and the output $Y(s)$. The time domain solution to Eq.(3) and equivalently Eq.(2), depends on the form of the input. Assume that the input is a unit step change in $u(t)$, i.e.

$$u(t) = 1$$

From Table 1, the Laplace Transform of $u(t)$ is:

$$L\{u(t)\} = L\{1\} = U(s) = 1/s$$

Substituting into Eq.(5),

$$Y(s) = \frac{K}{(1+\tau s)} \cdot \frac{1}{s} = K \cdot \frac{(1/\tau)}{s[(1/\tau)+s]} \qquad (6)$$

Looking up the inverse Laplace Transform of Eq.(6) from Table 1 yields the time domain solution of Eq.(4) as:

$$y(t) = K\left[1 - e^{-t/\tau}\right] \quad \text{or} \quad Y(t) = K\left[1 - e^{-t/\tau}\right] + Y_{ss} \qquad (7)$$

TRANSFER FUNCTIONS

Transfer functions play a central role in the analysis of dynamic systems behaviour since it fully describes the relation between input-output pairs. In the previous example, the transfer function $G(s)$ encapsulates the behaviour of the system given by Eq.(1) in the ratio, Eq.(5). There are several things to note about transfer functions.

Transfer functions are independent of the form of the input. If instead of a unit step input ($u(t) = 1$), a ramping input, $u(t) = t$ was considered, $G(s)$ would still describe the behaviour of the output $y(t)$. However, the time-domain solution would be different from that given by Eq.(7).

Transfer functions obey algebraic rules. Laplace Transforms are *commutative* as well as *associative*. Therefore, given two transfer functions $G_1(s)$ and $G_2(s)$:

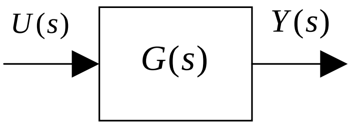
$G_1(s)G_2(s) = G_2(s)G_1(s)$ Commutative property

$G_1(s) + G_2(s) = G_2(s) + G_1(s)$ Associative property

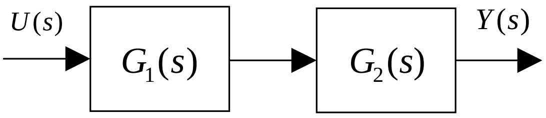
Transfer functions are linear functions. Since they are expressed in the Laplace domain, transfer functions are implicitly linear functions and obey all the rules governing the theory of linear systems.

Transfer functions can be visualised conveniently using block diagrams. Another useful feature of using Laplace Transforms is that the dynamical system can be expressed in the form of block diagrams. Each block represents a transfer function while the signal flows between the blocks are defined by block connections. As a result, the system can be easily visualised and relationships between inputs and outputs worked out via block diagram manipulations. Several examples are shown below:

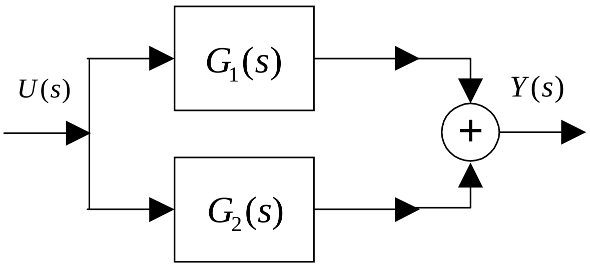
Block Diagram | **Transfer Function**



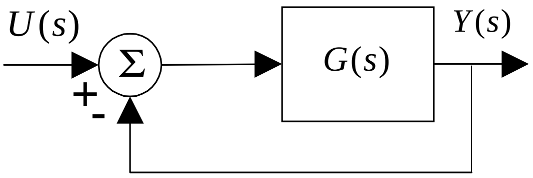
$$\frac{Y(s)}{U(s)} = G(s)$$



$$\frac{Y(s)}{U(s)} = G_1(s)G_2(s)$$



$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$



$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)}$$

Transfer functions are ratios of polynomials in 's'. From the above block diagrams, it can be seen that transfer functions can be made up of a combination of other transfer functions. Thus, in general, transfer functions are ratios of polynomials in '*s*' as illustrated in the following example. Suppose

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

with $G_1(s) = \dfrac{K_1}{1 + \tau_1 s}$ and $G_2(s) = \dfrac{K_2}{1 + \tau_2 s}$

Then $\dfrac{Y(s)}{U(s)} = \dfrac{K_1}{1 + \tau_1 s} + \dfrac{K_2}{1 + \tau_2 s}$

and hence

$$\frac{Y(s)}{U(s)} = \frac{K_1(1 + \tau_2 s) + K_2(1 + \tau_1 s)}{(1 + \tau_1 s)(1 + \tau_2 s)}$$

$$\frac{Y(s)}{U(s)} = \frac{(K_1 + K_2) + (K_1\tau_2 + K_2\tau_1)s}{(1 + \tau_1 s)(1 + \tau_2 s)}$$