32. Shannon Information Capacity Theorem and Implications

Shannon Information Capacity Theorem

Shannon's information capacity theorem states that the channel capacity of a continuous channel of bandwidth W Hz, perturbed by bandlimited Gaussian noise of power spectral density $n_0/2$, is given by

$$C_c = W \log_2\left(1 + \frac{S}{N}\right) \qquad \text{bits/s} \qquad (32.1)$$

where S is the average transmitted signal power and the average noise power is

$$N = \int_{-W}^{W} n_0/2 \, dw = n_0 W \qquad (32.2)$$

Proof [1].

Suppose that we transmit one of a set of M equiprobable signals of bandwidth W in time T. Each signal thus represents $\log_2 M$ bits. According to the sampling theorem, each signal can be represented by $n = 2WT$ samples in T seconds. Assume that the maximum average signal power is S and the noise power is N. In the geometrical representation, all the transmitted signals must be restricted to an n-dimensional hypersphere of radius $ST^{0.5}$ around the origin corresponding to their maximum energy. Similarly, all the received signals are restricted to an overall signal space of radius $[(S + N)T]^{0.5}$. This is shown in Figure 32.1.

Figure 32.1 Signal space for calculating channel capacity.

A noise power greater than NT will cause incorrect detection. In the presence of noise, the channel capacity can be determined by the number of signals that can be accommodated in the signal space.

The volume of an n-dimensional hypersphere is proportional to r^n, where r is the radius of the hypersphere. Hence the number of signals that can be accommodated in an n-dimensional signal space is

$$\begin{aligned} M &\le [(S + N)T]^{0.5n} / (NT)^{0.5n} \\ &\le [1 + (S/N)]^{0.5n} \end{aligned}$$

The information per signal is

$$\log_2 M \le \frac{n}{2}\log_2[1 + (S/N)]$$

and the channel capacity is

$$\begin{aligned} C_c &= \frac{1}{T}\log_2 M \\ &\le \frac{n}{2T}\log_2[1 + (S/N)] \\ &\le W\log_2[1 + (S/N)] \qquad \square \end{aligned}$$

Example 32.1

If W = 3 kHz and S/N is maintained at 30 dB for a typical telephone channel, the channel capacity C_c is about 30 kbits/s.

The theorem implies that error-free transmission is possible if we do not send information at a rate greater than the channel capacity. Thus, the information capacity theorem defines the fundamental limit on the rate of error-free transmission for a power limited, bandlimited Gaussian channel.

Figure 32.2 shows the general form of encoding scheme suggested by Shannon. A binary sequence of length R_b bits in a second are encoded into a binary sequence of length $R_b\ T_b$ bits in T_b seconds before transmission. However, the design of the encoder and decoder is left unspecified.

Figure 32.2 Error-free transmission system model.

It can be seen that the encoding time is T_b seconds. There is a encoding delay of T_b seconds in transmission and a decoding delay of T_b seconds at the receiver. A total delay of 2 T_b seconds is entailed. We can reduce the delay by decreasing the value of T_b, but we require more channel bandwidth for transmission.

In the case of no bandwidth limitation, it can be shown that the channel capacity approaches a limiting value C_∞ given by

$$C_\infty = \lim_{W \to \infty} C_c = \frac{S}{n_0}\log_e 2 = 1.44\ \frac{S}{n_0} \qquad (32.3)$$

The channel capacity variation with bandwidth is shown in Figure 32.3.

Figure 32.3 Channel capacity variation with bandwidth.

Proof.

$$
\begin{aligned}
C_c &= W \log_2(1 + S/N) \\
&= W \log_2(1 + \frac{S}{n_0 W}) \\
&= (\frac{S}{n_0})(\frac{n_0 W}{S}) \log_2(1 + \frac{S}{n_0 W}) \\
&= (\frac{S}{n_0}) \log_2[(1 + \frac{S}{n_0 W})^{(n_0 W/S)}]
\end{aligned}
$$

Since $\lim_{x \to 0} (1 + x)^{1/x} = e$, we have

$$C_c = \frac{S}{n_0} \log_2 e = 1.44 \frac{S}{n_0} \qquad \square$$

Implications [2, 3]

1. Capacity of M-point QAM Signals

In bandlimited channels, how does the capacity of M-point QAM signals compare to Shannon's information capacity limit? In this example, we derive the capacity of M-ary QAM signals. Assume that each M-point QAM signal symbol has a duration of T seconds. We can represent each M-point QAM signal by $\log_2 M$ bits. Thus, we have

$\log_2 M$ bits/symbol,
$1/T$ symbols/s,

and the transmission rate R_b is

$$R_b = (\log_2 M)/T \qquad \text{bits/s} \qquad (32.4)$$

Suppose that the bandwidth of the M-ary QAM signals is set equal to the channel bandwidth W. Using the definition of the null-to-null bandwidth, the bandwidth of the M-ary QAM signals is $W = (f_c + \frac{1}{T}) - (f_c - \frac{1}{T}) = 2/T$, where f_c is the carrier frequency. Hence, we may express the transmission rate of equation (32.4) as

$$R_b = \frac{W}{2} \log_2 M \qquad \text{bits/s} \qquad (32.5)$$

For a fixed spacing between adjacent signals, increasing the value of M also increases the average transmitted signal power S. Accordingly, we increase the signal-to-noise ratio. Let $M = K'\frac{S}{N}$, where K' varies with error rate and is a constant small enough to achieve negligible error rate. We have

$$R_b = \frac{W}{2}\log_2\left(K'\frac{S}{N}\right) \qquad \text{bits/s} \qquad (32.6)$$

The capacity of an M-ary QAM system approaches the Shannon channel capacity C_c if the average transmitted signal power in the QAM system is increased by a factor of $1/K'$.

The Shannon information capacity theorem tells us the maximum rate of error-free transmission over a channel as a function of S, and equation (32.6) tells us what is achievable for a practical M-ary QAM system.

2. Capacity of an n-ary PCM system

In this example, we derive the capacity of an n-ary PCM system. Assume that an input analogue signal of bandwidth W Hz is sampled at the minimum Nyquist sampling rate of $2W$ samples/s and the samples are uniformly quantised to $M = n^m$ levels.

We can represent each M-level signal sample by m n-ary symbols. This is shown in Figure 32.4.

Figure 32.4 Representations of quantised sample.

Thus we have

$2W$ samples/s,
$M = n^m$ levels/sample,
m symbols/sample,
$\log_2 n$ bits/n-ary symbol, and
$m \log_2 n$ bits/sample.

The symbol rate is $2W\, m$ symbol/s and the information transmission rate is

$$R_b = 2W\, m \log_2 n \qquad \text{bits/s} \qquad (32.7)$$

For error-free transmission, the channel capacity $C_c \geq R_b$.

Observation: For fixed values of n and m, the capacity R_b is proportional to W.

Let S be the average transmitted signal power and a be the spacing between n-levels. We assume that the n discrete levels are equally likely and have the values $\pm a/2$, $\pm 3a/2$, ..., $\pm (n-1)a/2$. The average transmitted signal power is

$$\begin{aligned} S &= (1/n)\{(a/2)^2 + (3a/2)^2 + \ldots + [(n-1)a/2]^2\} \times 2 \\ &= a^2(n^2 - 1)/12 \end{aligned} \qquad (32.8)$$

Expressing n in terms of S and substituting into (32.7), we get

$$R_b = W \log_2 \left(1 + \frac{12S}{a^2}\right) \qquad (32.9)$$

To maintain a negligible error rate, there must be a finite separation a between adjacent n-ary levels. Call this separation $a = K\sigma$, where K varies with the error rate and is a constant large enough to allow recognition of individual levels with negligible error rate, and $\sigma^2 = N$ is the noise power. We have

$$R_b = W \log_2 \left(1 + \frac{12S}{K^2 N}\right) \quad \text{bits/s} \qquad (32.10)$$

Observation: We can trade-off bandwidth for signal-to-noise ratio for a system with given channel capacity $C_c = R_b$.

The capacity expression of an n-ary PCM system is identical to the Shannon channel capacity expression if the average transmitted signal power in the PCM system is increased by a factor of $K^2/12$.

References

[1] Burr, A., Modulation and Coding for Wireless Communications, Pearson Education, 2001.

[2] Haykin, S., Communication Systems, 4/e, J. Wiley & Sons, 2001.

[3] M. Schwartz, Information Transmission, Modulation, and Noise, 4/e, McGraw-Hill, 1990.

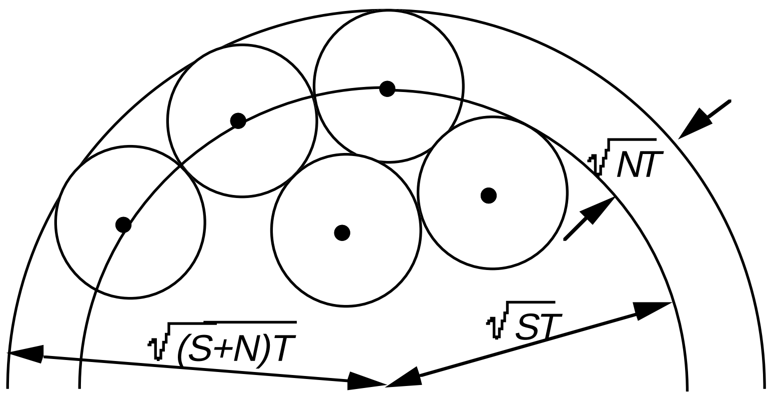

Figure 32.1 Signal space for calculating channel capacity.

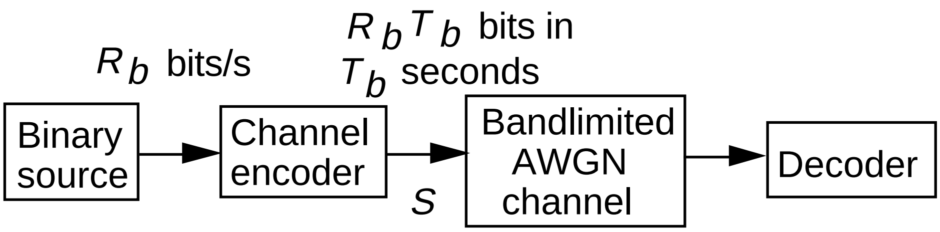

Figure 32.2 System model to achieve error-free transmission.

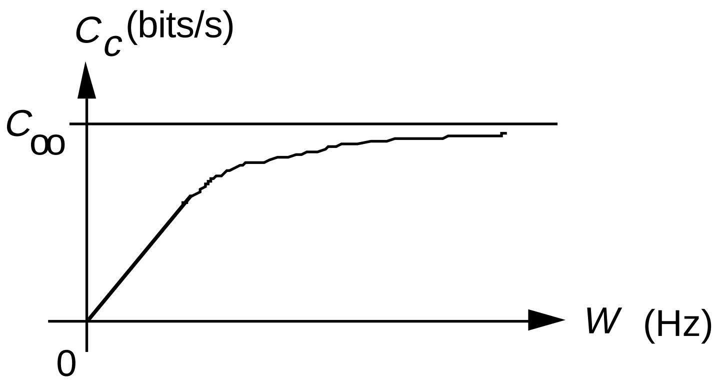

Figure 32.3 Channel capacity variation with bandwidth.

$M = n^m$ Total no. of amplitude levels

m - No. of pulses per sample

n - No. of possible amplitude levels per pulse

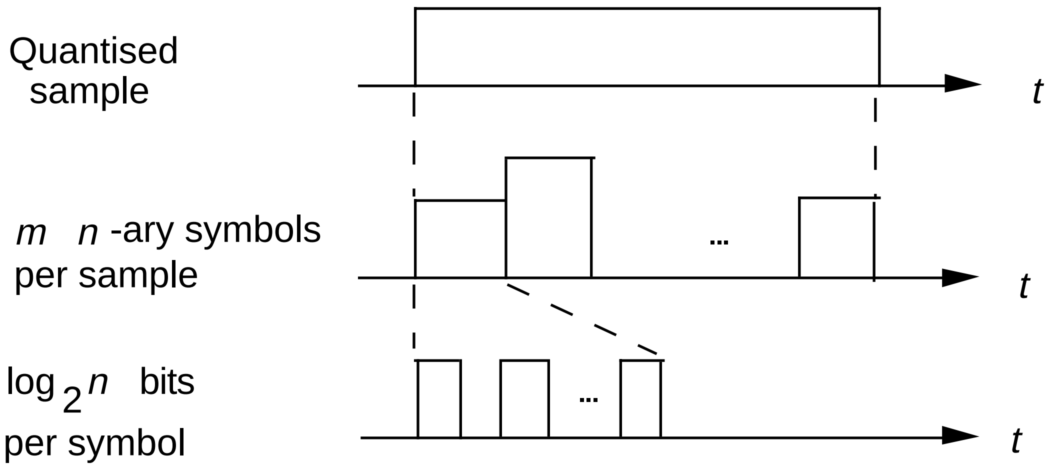

Figure 32.4 Representations of quantised sample.