ELEC1011 Communications and Control

(5/11) Phasors and Constellation Diagrams

Multiplication of two sinusoids (CEP 2.3.4)

Trigonometric identity (1) from Lecture 3 shows what happens when you multiply two sinusoids having different frequencies

$$\begin{aligned} v(t) &= A_1 \cos(2\pi f_1 t + \phi_1) \cdot A_2 \cos(2\pi f_2 t + \phi_2) \\ &= \frac{A_1 A_2}{2} \cos(2\pi [f_1 - f_2] t + \phi_1 - \phi_2) + \frac{A_1 A_2}{2} \cos(2\pi [f_1 + f_2] t + \phi_1 + \phi_2) \end{aligned} \quad (1)$$

It also shows what happens when you multiply two sinusoids having the same frequency

$$\begin{aligned} v(t) &= A_1 \cos(2\pi f t + \phi_1) \cdot A_2 \cos(2\pi f t + \phi_2) \\ &= \frac{A_1 A_2}{2} \cos(\phi_1 - \phi_2) + \frac{A_1 A_2}{2} \cos(4\pi f t + \phi_1 + \phi_2) \end{aligned} \quad (2)$$

Addition of two sinusoids (CEP 2.3.3)

The addition of two sinusoids having different frequencies cannot be simplified any further

$$v(t) = A_1 \cos(2\pi f_1 t + \phi_1) + A_2 \cos(2\pi f_2 t + \phi_2)$$

The addition of two sinusoids having the same frequency can be simplified further

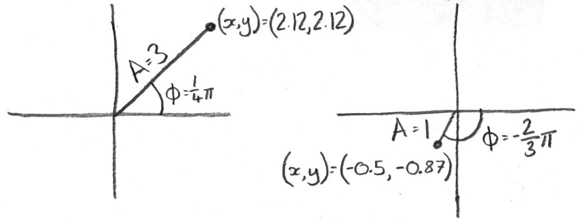
$$\begin{aligned} v(t) &= A_1 \cos(2\pi f t + \phi_1) + A_2 \cos(2\pi f t + \phi_2) \\ &= A \cos(2\pi f t + \phi) \end{aligned}$$

The result is a single sinusoid having the same frequency, but a new amplitude and phase.

Phasors (CEP 2.3.3.2)

e.g. $v(t) = 3\cos(2\pi f t + \frac{1}{4}\pi) + \cos(2\pi f t - \frac{2}{3}\pi)$

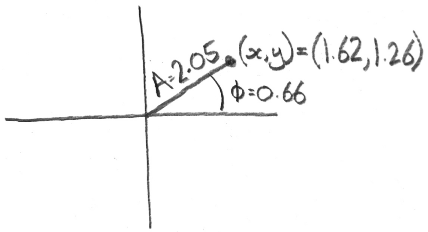
- Represent each sinusoid with a phasor



- Determine the coordinates (x,y) of each phasor using trigonometry

  $x = A\cos(\phi) \quad y = A\sin(\phi)$

- Add the coordinates to get the resultant phasor



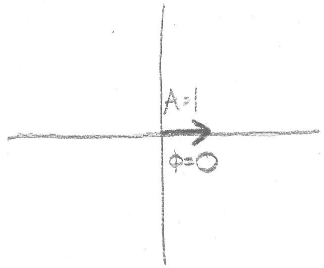
- Determine the amplitude A and phase ϕ of the phasor

$$A = \sqrt{x^2 + y^2} \qquad \phi = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \end{cases}$$

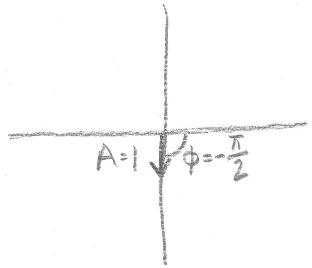
so $v(t) = 2.05\cos(2\pi f t + 0.66)$

Phasors continued

$\cos(2\pi f t)$



$\sin(2\pi f t) = \cos(2\pi f t - \frac{\pi}{2})$

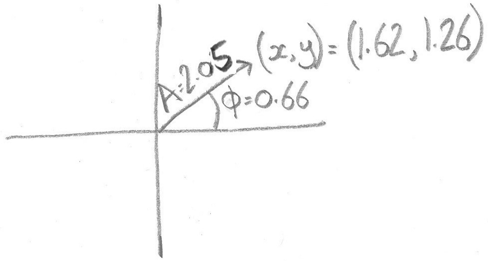


Any phasor can be thought of as a sum of a cos and a sin

$$A\cos(2\pi f t + \phi) = x\cos(2\pi f t) - y\sin(2\pi f t)$$

where $x = A\cos(\phi)$ and $y = A\sin(\phi)$ as before

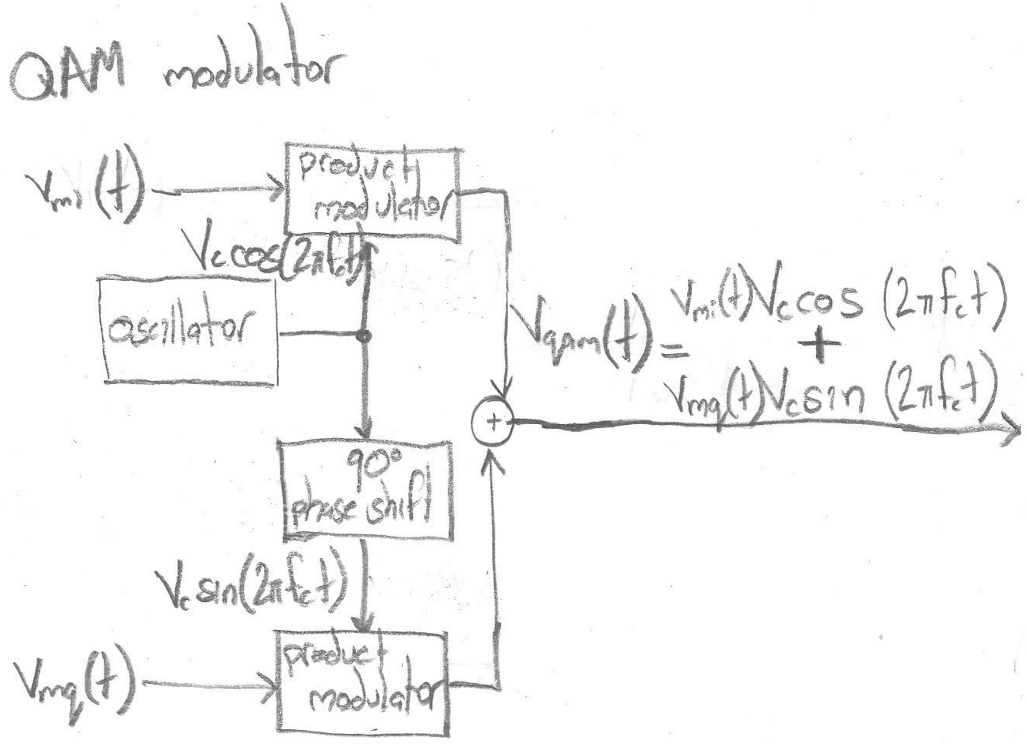
eg $v(t) = 2.05\cos(2\pi f t + 0.66)$ from before



$$v(t) = 1.62\cos(2\pi f t) - 1.26\sin(2\pi f t)$$

QAM revisited

- QAM uses an in-phase carrier $V_c \cos(2\pi f_c t)$ and a quadrature-phase carrier $V_c \sin(2\pi f_c t)$ having the same frequency.



- The x coordinate of a phasor is equivalent to the in-phase signal $v_{mi}(t)$ of QAM.
- The y coordinate of a phasor is equivalent to the quadrature-phase signal $-v_{mq}(t)$.

Constellation diagrams (CEP 7.9.2, 7.9.3 and 7.9.5)

- In Lecture 2, we quickly looked at ASK, PSK and FSK digital carrier modulation schemes.

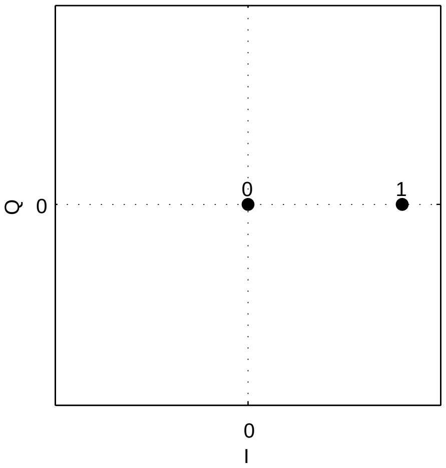
- For ASK and PSK, phasors can be drawn in a constellation diagram to show how the amplitude and phase of the carrier is modulated in order to signal different combinations of bit values.

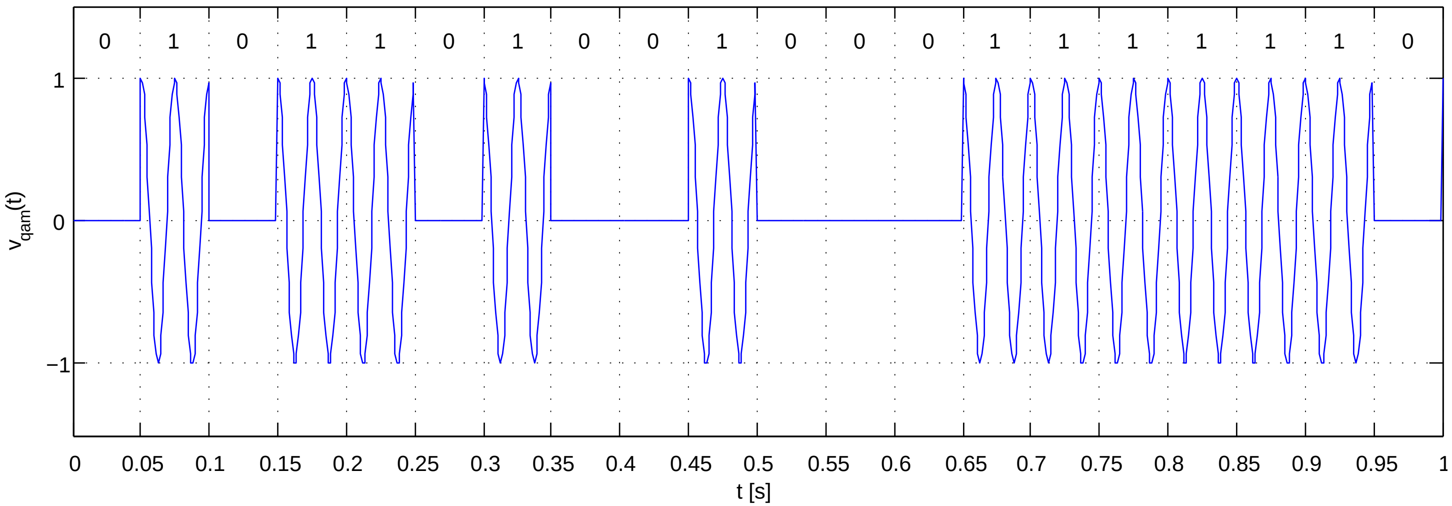
- Each of the $M = 2^k$ constellation points is labelled with a different combination of k bits.

- The constellation points can be positioned and labelled in many different ways. Here are some examples...

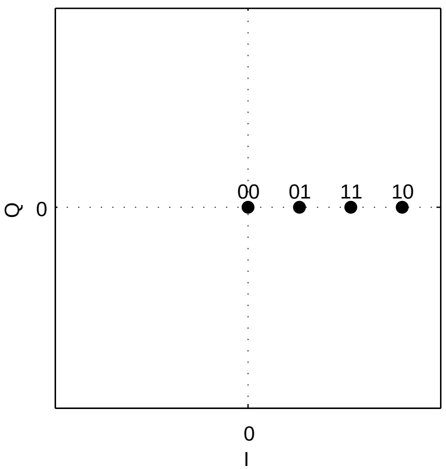
Constellation diagrams (CEP 7.9.2)

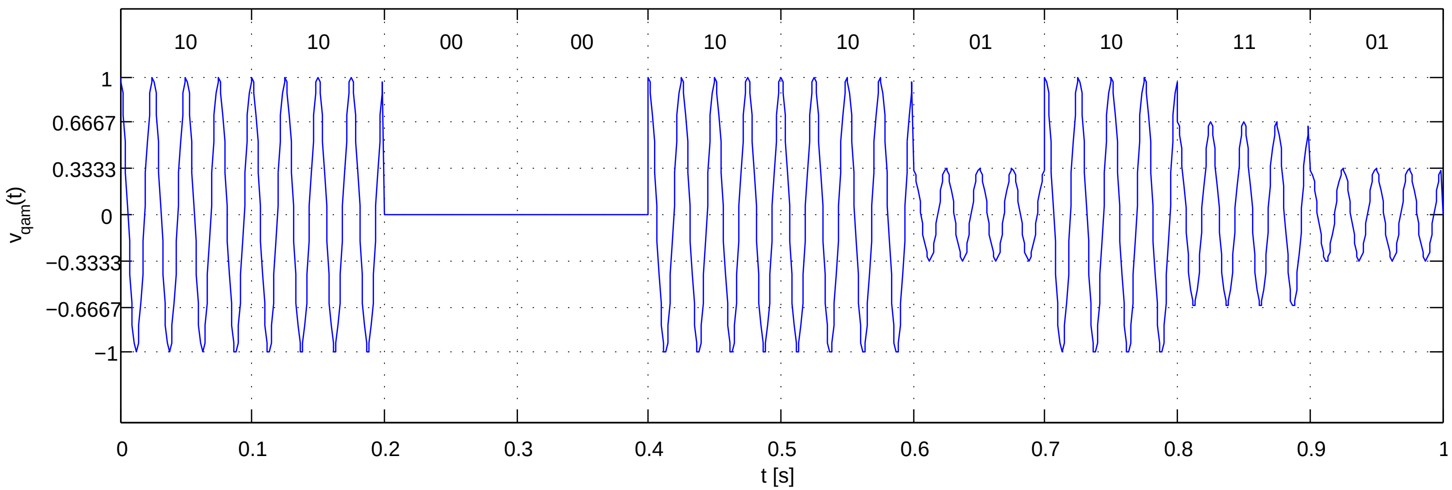
- On-Off Keying (OOK)





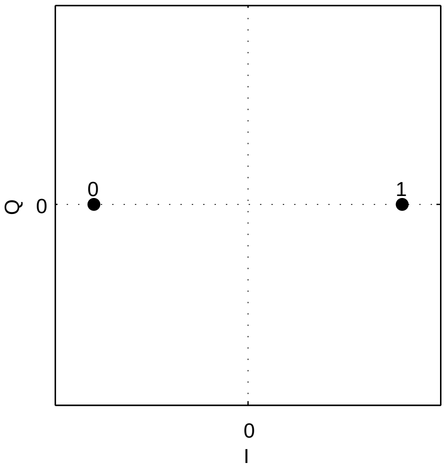
- $M = 4$-ary Amplitude Shift Keying (4ASK)

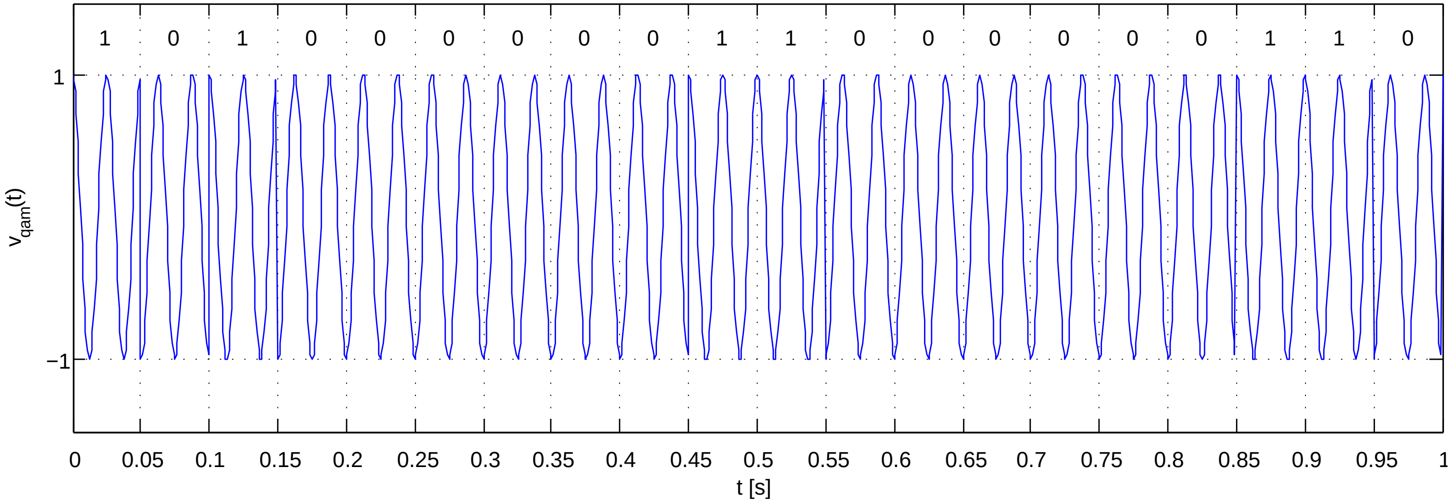




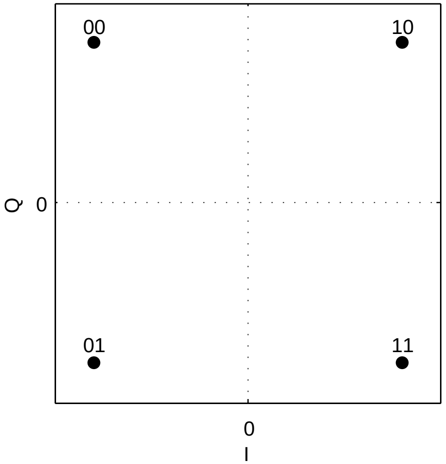
Constellation diagrams (CEP 7.9.3)

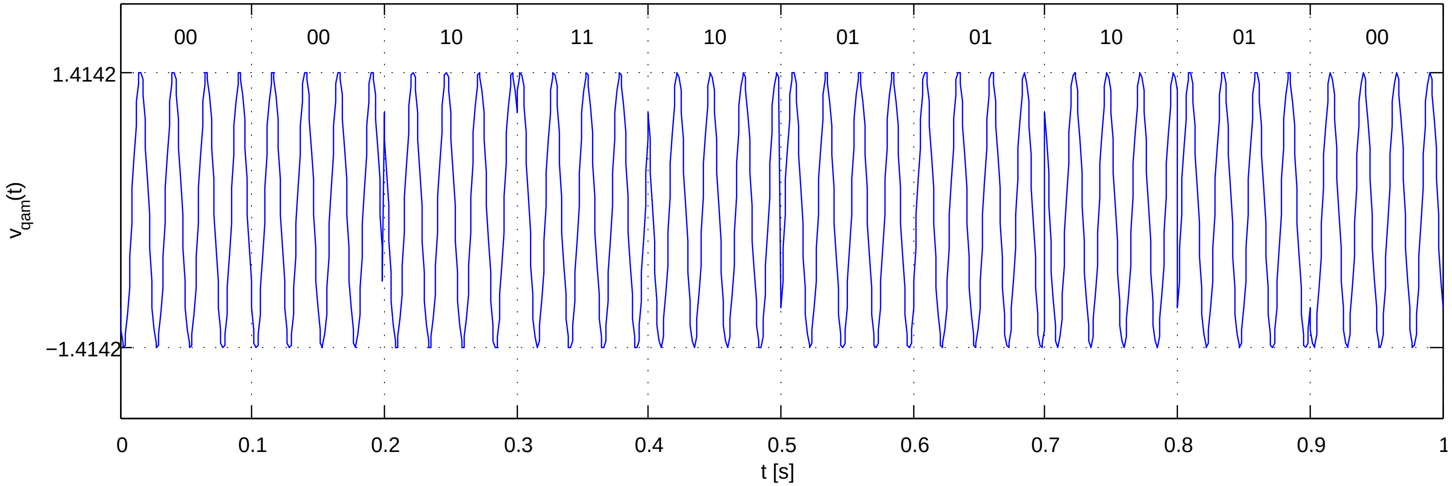
- Binary ($M = 2$) Phase Shift Keying (BPSK)





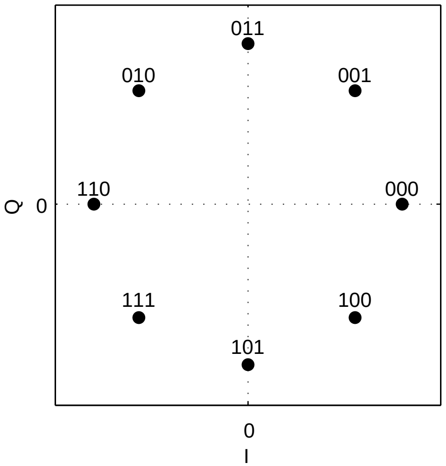
- Quarternary ($M = 4$) Phase Shift Keying (QPSK)

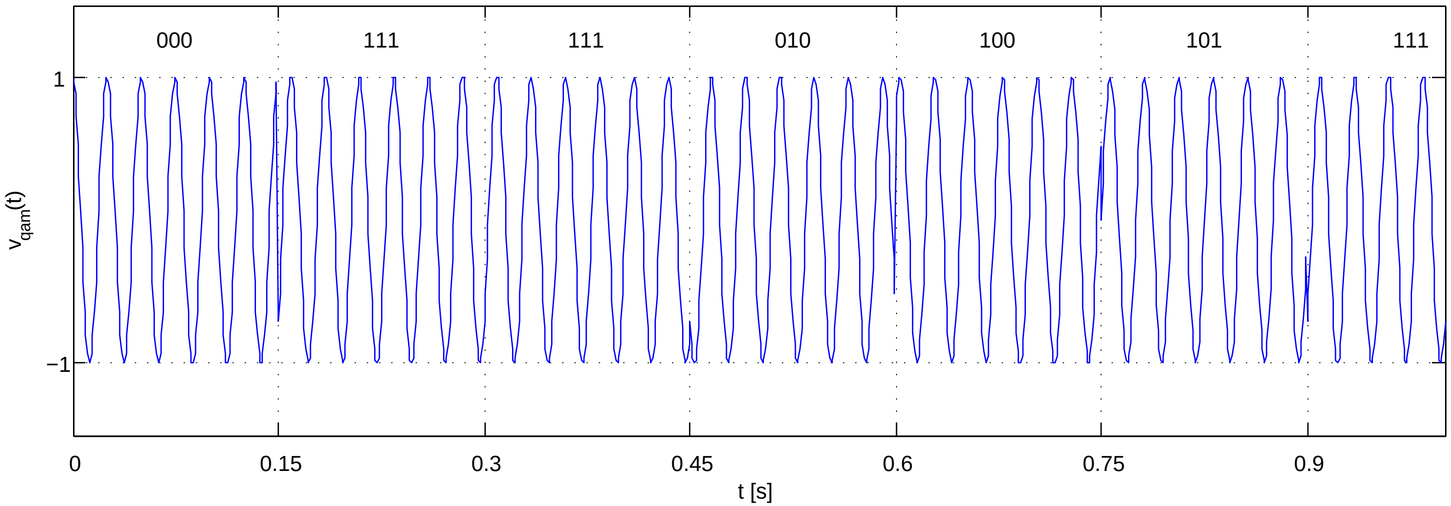




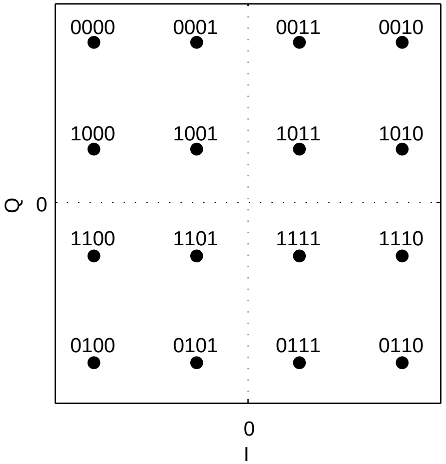
Constellation diagrams (CEP 7.9.3 and 7.9.5)

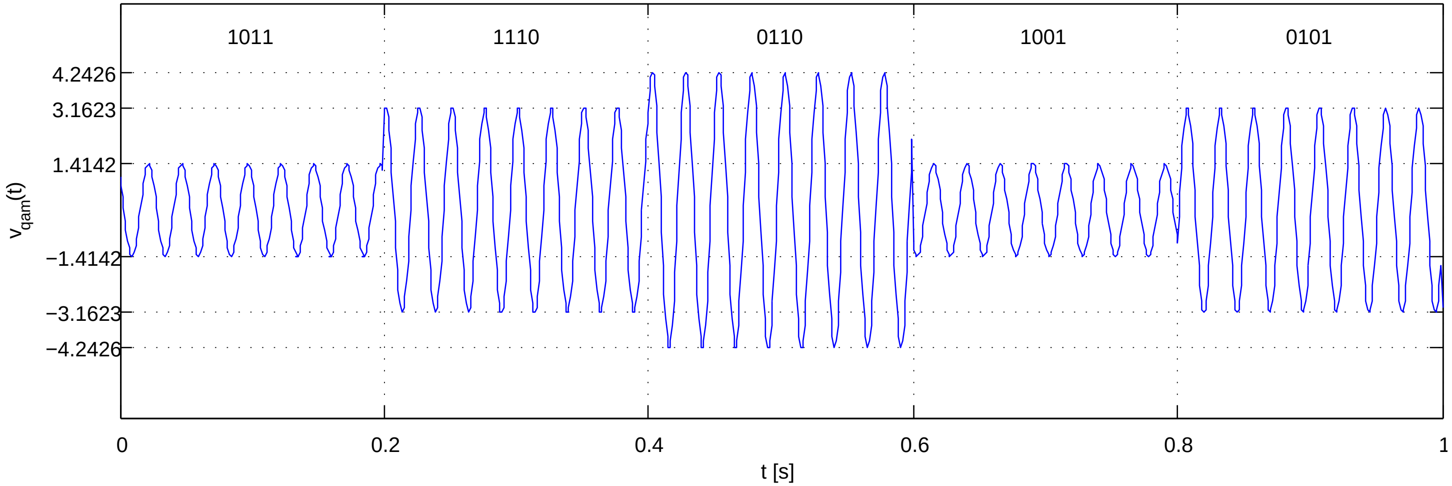
- $M = 8$-ary Phase Shift Keying (8PSK)



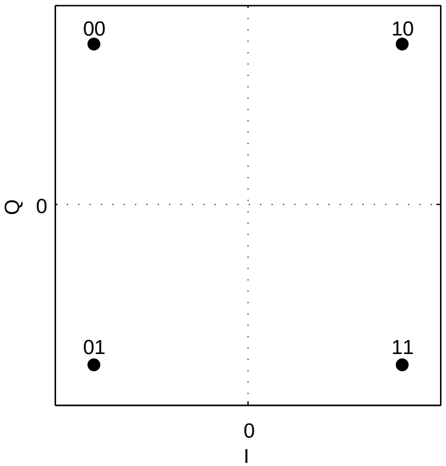


- $M = 16$-ary Quadrature Amplitude Modulation (16QAM)

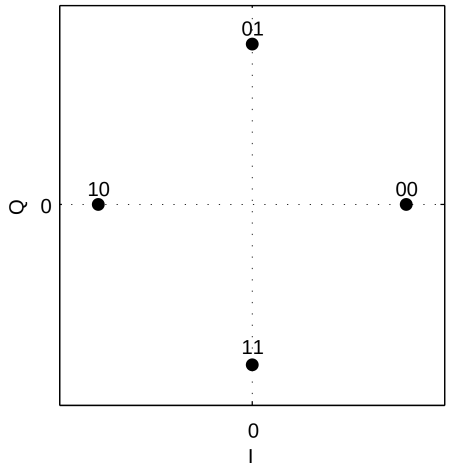




Different positioning and labellings of constellation points



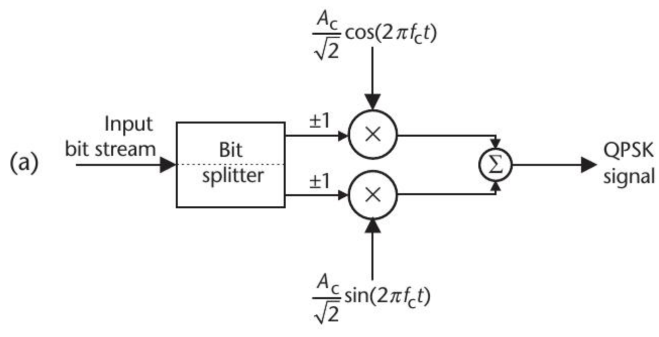
- The QPSK constellation diagram on the left is the one from just now.
- This differs to that of the QPSK scheme described in Lecture 2, which is shown on the right.



The left-hand scheme has two advantages:

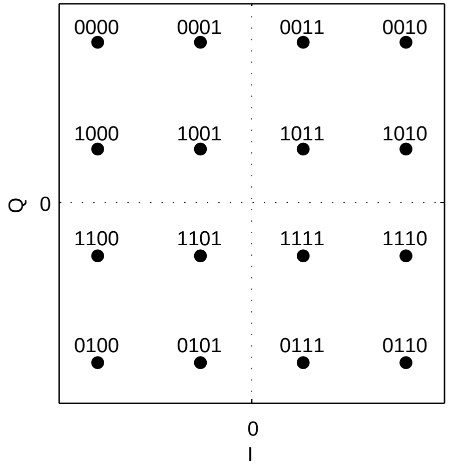
- It is simpler to implement using a QAM modulator, since the in-phase signal depends only on the first bit and the quadrature-phase signal depends only on the second bit.
- It uses Gray coding, which means that the labels of adjacent constellation points differ by only one bit.

Figure 7.41



Gray coding (CEP 7.9.2.1)

• Gray coding is easiest to see when it is applied to 16QAM.
• 16QAM is a cross between ASK and PSK, since it modulates both the amplitude and the phase of the carrier.
• 16QAM is less sensitive to noise than 16ASK and 16PSK because its constellation points are more spread out.



• Gray coding further reduces 16QAM's sensitivity to noise by labelling adjacent constellation points with bit sequences that differ by only one bit.

• This is advantageous because, the closer two constellation points are, the more likely it is for noise to cause them to be confused with each other.

• As a result, the most likely effect of noise is to cause only one bit error.

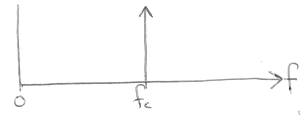
Gray coding (CEP 7.9.2.1)

A sequence of Gray coded bit sequences can be obtained by starting with a sequence of all zeros. Each subsequent sequence is obtained by toggling the bit in the previous sequence that is furthest to the right without repeating a previous sequence.

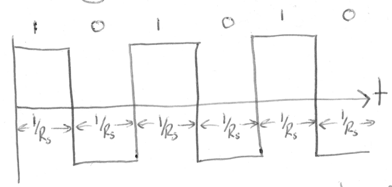
k	$M = 2^k$	Gray code
1	2	0 1
2	4	00 01 11 10
3	8	000 001 011 010 110 111 101 100
4	16	0000 0001 0011 0010 0110 0111 0101 0100 1100 1101 1111 1110 1010 1011 1001 1000
5	32	00000 00001 00011 00010 00110 00111 00101 00100 01100 01101 01111 01110 01010 01011 01001 01000 11000 11001 11011 11010 11110 11111 11101 11100 10100 10101 10111 10110 10010 10011 10001 10000

Spectrum of digital carrier modulation (CEP 7.6.4)

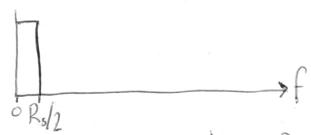
ASK, PSK and digital-QAM use a carrier having a constant frequency f_c so you may think the spectrum looks like this..



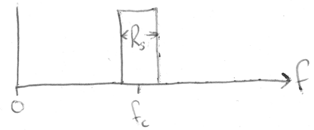
However, the amplitude and phase of the carrier are switched at the symbol rate R_s. In the worst case scenario, the carrier will be switched on every symbol like this...



This square wave has a period of $2/R_s$ and a frequency of $R_s/2$. It's spectrum (when filtered) looks like...



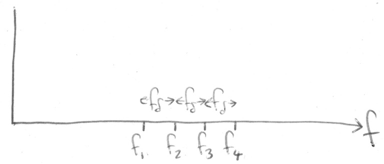
Therefore, the spectrum of an ASK, PSK and digital-QAM signal looks like...



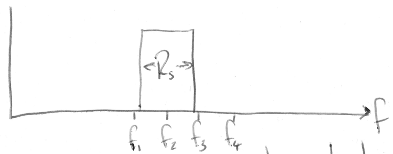
The bandwidth is R_s irrespective of how many bits per symbol are transmitted.

Spectrum of M-ary FSK

M-ary FSK activates one of M carrier frequencies, depending on the values of the $k = \log_2(M)$ bits. The carrier frequencies are spaced f_δ apart. For example, in 4-ary FSK...

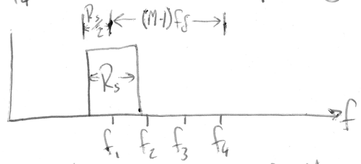


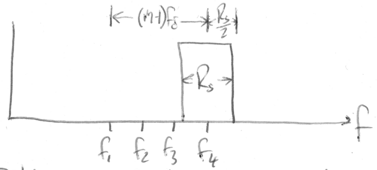
When a carrier is activated, the bandwidth surrounding it is given by the symbol rate, as in ASK, PSK and digital-QAM. For example, when f_2 is activated...



In order to ensure that activating f_2 does not also activate the neighbouring frequencies f_1 and f_3, we need $f_\delta \geq R_s/2$.

The lowest and highest frequencies that are activated occur when f_1 and f_4 are activated, respectively...





The total bandwidth of M-ary FSK is given by

$$(M-1)f_\delta + R_s/2 + R_s/2 \geq (M+1)R_s/2$$

Pros and cons of ASK, PSK, digital-QAM and FSK (CEP 7.9.2 – 7.9.5)

- In all schemes, noise can cause bit errors in the receiver.
- The Bit Error Ratio (BER) depends on the ratio of the transmission power to the channel's noise power.
- In ASK, PSK and digital-QAM, the transmission power per bit required to achieve a particular low BER increases with M.
- In FSK, the transmission power per bit required to achieve a particular low BER reduces with M.
- Hence, FSK is more power efficient than ASK, PSK and 16QAM.
- 16QAM requires a lower transmission power than 16PSK, which requires a lower transmission power than 16ASK.
- ASK can be demodulated using a simple envelope detector, whereas PSK and digital-QAM require coherent detection.
- PSK is immune to amplitude fading, whereas digital-QAM is not.

Bandwidth efficiency of ASK, PSK, digital-QAM and FSK (CEP 7.9.1)

- The ratio of the bit rate R_b to the bandwidth required B is referred to as the bandwidth efficiency $\eta = R_b/B$.
- For all schemes, the bit rate R_b is related to the symbol rate R_s according to $R_b = kR_s = \log_2(M)R_s$.
- The bandwidth required by ASK, PSK and digital-QAM is related to the symbol rate according to $B = R_s$ (it does not depend on M).
- So, the bandwidth efficiency of ASK, PSK and digital-QAM is given by $\eta = \log_2(M)$, which increases with M.
- The bandwidth required by FSK is related to the symbol rate R_s according to $B = R_s(M+1)/2$.
- So, the bandwidth efficiency of FSK is given by $\eta = \frac{2\log_2(M)}{M+1}$, which decreases with M.
- Hence, ASK, PSK and digital-QAM are more bandwidth efficient than FSK.

Bandwidth efficiency of digital and analogue communications

- Audio has a bandwidth of $f_{max} = 20$ kHz and is typically sampled at 44.1×10^3 samples per second.
- 32 bits per sample are typically used for stereo, giving a bit rate of $R_b = 1411.2$ kbps.
- Using BPSK ($M = 2$) there is $k = \log_2(M) = 1$ bit per symbol, giving a symbol rate and signal bandwidth of $B = R_s = R_b/k = 1411.2$ kHz.
- Similarly, $M = 4$, $M = 8$ and $M = 16$-ary digital modulation would give bandwidths of 705.6 kHz, 470.4 kHz and 352.8 kHz, respectively.
- By contrast, Double SideBand Suppressed Carrier (DSBSC) analogue modulation requires a bandwidth of only $2f_{max} = 40$ kHz.
- To achieve this bandwidth, $M = 2^{32} = 4 \times 10^9$-ary digital modulation would be required!
- Alternatively, lossy digital compression techniques such as MP3 can be used to reduce the bit rate to 128 kbps, requiring only $M = 8$-ary digital modulation.

Exercise

1. Draw phasor diagrams for the sinusoids $3\cos(200\pi t + \pi/4)$ and $-2\sin(200\pi t)$.

2. Express the sum of these sinusoids as a single sinusoid and draw the corresponding phasor diagram.

3. Express the new sinusoid as the sum of in-phase and quadrature-phase sinusoids.

4. Draw a constellation diagram for Gray-coded $M = 8$-ary ASK.

5. Draw a time-domain waveform of the signal that results when the bit sequence 001101110111100 is $M = 8$-ary ASK modulated onto a $f_c = 100$ Hz carrier using a symbol rate of $R_s = 20$ Hz.

6. What range of frequencies does the resultant ASK signal occupy? What is the bandwidth of this signal?

7. What is the minimum bandwidth that would be required for the corresponding FSK signal?