Differential Equations

INTRODUCTION

It often happens that in creating a mathematical model of a physical system, we need to express such relationship as 'the acceleration of A is directly proportional to B' or 'changes in D produce proportionate changes in E with constant of proportionality F'. Such statements naturally give rise to equations involving derivatives and integrals of the variables in the mathematical model as well as the variables themselves.

Equations which introduce derivatives are called *differential equations.*

MODELLING EXAMPLES

Population growth

A population grows at a rate, which is equal to the present population, i.e. the more organisms present, the faster the population grows.

In mathematical form, the population $y(t)$ is a function of time *t* and the rate of growth is $y'(t) = \frac{dy}{dt}$.

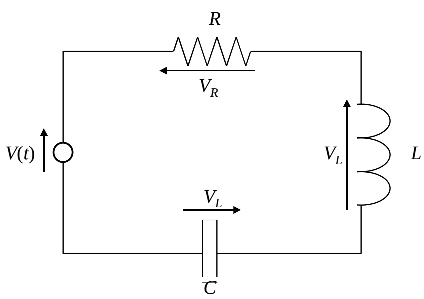
The above definition means that

$$\frac{dy}{dt} = y(t)$$

This differential equation models the system. Solving the differential equation would provide $y(t)$ a function of time that satisfies it.

LCR-electrical circuit

A resistor, a capacitor and an inductor are connected in a serie circuit.



- the voltage across the resistor is: $V_R = RI$.
- the voltage across the inductor is: $V_L = L\frac{dI}{dt}$.
- the voltage across the capacitor is: $V_C = \frac{q}{C}$, where *q* is the electrical charge in the capacitor.

The principle of conservation of charge tells us that the current flowing through the capacitor is equal to the rate of change of charge, that is we have,

$$I = \frac{dq}{dt} \quad \Leftrightarrow \quad q = \int I dt$$

The Kirchoff's voltage law states that the voltage around the circuit (or a loop) must be zero, so that our model is $V_L + V_R + V_C = V(t)$. Thus,

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V(t)$$

Here, we have two options. Either we eliminate *q*, in which case we obtain the *integro-differential equation*:

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int I dt = V(t)$$

or we eliminate *I*, in which case we obtain the differential equation:

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

These equations are of course, equivalent, but the final form is probably the most usual and tractable of the three. Thus, we have found that a simple analysis of the LCR electrical circuit results in a differential equation for one of the circuit variables: either the charge on the capacitor or the current in the circuit.

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

When creating mathematical models of problems chosen from different areas of engineering, each give rise to a differential equation. There are many techniques for solving differential equations - different methods being applicable to different kinds of equation. So, it is necessary to understand the various categories and classifications of differential equations.

Independent and dependent variables

The next classification you must understand is that of the variables occurring in a differential equation.

The variables with respect to which differentiation occurs are called *independent variables*, while variables/functions that are differentiated are *dependent variables*.

Order of differential equations

Another classification of differential equations is in terms of their order.

The order of an ordinary differential equation is the degree of the highest derivative that occurs in the equation.

The order of a differential equation *is not affected by any power to which the derivative may be raised*.

Examples:

(a) $$\frac{d^2 f(x)}{dx^2} - 4x\frac{df(x)}{dx} = \cos 2x$$

(b) $$\begin{cases} 4\dfrac{dx}{dt} + 3\dfrac{dy}{dt} - x + 2y = \cos t \\ 6\dfrac{dx}{dt} - 2\dfrac{dy}{dt} - 2x + y = 2\sin t \end{cases}$$

(c) $$\left(\frac{dx}{dt}\right)^2 + 4\frac{dx}{dt} = 0 .$$

Linear and non-linear differential equations

Differential equations are also classified as linear and non-linear. We may informally define *linear equations as those in which the dependent variables and their derivatives do not occur as product, raised to powers or in non-linear functions.*

Non-linear equations are those that are not linear. Linear equations are an important category, since they have useful simplifying properties.

Homogeneous and inhomogeneous equations

There is a further classification that can be applied to linear differential equations: the distinction between homogeneous and inhomogeneous equations.

We have presented so far differential equation that have been arrange so that all terms containing the dependent variables occur on the left-hand side and those terms that involve only independent variables and constant terms occur on the right-hand side. This is standard way of arranging terms, and aids in the identification of equations.

Specifically, when linear equations are arranged in this way, those in which *the right-hand side is zero, are called homogeneous equation* and those in which *it is non-zero are called inhomogeneous equations*.

The equations $\frac{dx}{dt} + 4x = 0$ and $4\frac{dx}{dt} + (\sin t)x = 0$ are both homogeneous differential equations.

While $\frac{d^2x}{dt^2} + x\frac{dx}{dt} = 4\sin t$ and $\frac{d^2 f}{dx^2} - 4x\frac{df}{dx} = \cos 2x$ are both inhomogeneous equations.

General and particular solutions

A differential equation will normally have many possible solutions, e.g. $y' = \cos x$ is a first-order linear differential equation. It can be solve by direct integration:

$$\int \frac{dy}{dx}dx = \int \cos x dx$$
$$y = \sin x + C$$

where C is an arbitrary constant, which disappears on differentiating.

The family of curves defined by $y = \sin x + C$ is *a general solution*. A specific value for C defines *a particular solution.* In simple cases, the particular solution is calculated from initial conditions (*initial value problem*).

A solution is a function, which satisfies the differential equation, i.e. if substituted, the differential equation becomes an *identity*.

E.g., verify that $y = x^2$ is a particular solution of $xy' = 2y$ for all x.

Differentiate y gives, $y' = 2x$,

substituting gives, $x \times 2x = 2x^2$: identity

Always check by substitution.

Solving 1st-order differential equations

DIRECT INTEGRATION

If a 1st-order differential equation can be arranged in the form:

$$\frac{dy}{dx} = f(x)$$

it can be solved by direct integration.

Example: Solve (worked during lecture)

$\frac{dy}{dx} - 3x^2 - 6x = 5$ and $x\frac{dy}{dx} = 5x^3 + 4$

INITIAL VALUE PROBLEM

In general, once the general solution is determined, we need to find the constant within the general solution, i.e. a particular solution.

Example: Solve,

$$e^x\frac{dy}{dx} = 4 \tag{1}$$

with initial condition $y = 3$ when $x = 0$.

1) Find the general solution of the differential equation (1).
2) Find a particular solution, i.e. determine *the constant term* from the initial condition so that the particular solution satisfies the differential equation.

SEPARABLE 1ST-ORDER DIFFERENTIAL EQUATIONS

It is not always possible to solve a differential equation by direct integration. Indeed, a more common 1st-order differential equation is

$$\frac{dy}{dx} = f(x,y) \text{ rather than } f(x).$$

If the function $f(x,y)$ is such that the differential equation can be manipulated (by algebraic operations) into the form:

$$g(y)dy = h(x)dx$$

then the differential equation is called *separable differential equation* and integrating both sides gives

$$\int g(y)dy = \int h(x)dx + C$$

i.e. *if we can write the differential equation with the x's and the y's on different side, we can solve it by separating variables.*

Example: Solve,

(a) $\frac{dy}{dx} = \frac{2x}{y+1}$ (b) $\frac{dy}{dx} = (1+x)(1+y)$

WORK EXERCISE 1

Find the general solution of the following differential equations:

(a) $\frac{dy}{dx} = \frac{1+y}{2+x}$ (b) $\frac{dx}{dt} = 4xt \quad , x > 0$

(c) $\frac{dx}{dt} = \frac{kx}{t}$ (d) $\frac{dx}{dt} = xte^{t^2}$

(e) $\frac{dx}{dt} = ax(x-1)$ (f) $x\frac{dx}{dt} = \sin t$

(g) $x^2\frac{dx}{dt} = e^t$ (h) $\frac{dx}{dt} = \frac{a}{xt}$

WORK EXERCISE 2

Find the particular solution of the following initial value problems:

(a) $\frac{dx}{dt} = \frac{\sin t}{x^2} \quad , x(0) = 4$

(b) $\frac{dy}{dt} = \frac{t^2+1}{y+2} \quad , y(0) = -2$

(c) $t(t-1)\frac{dx}{dt} = x(x+1), \quad x(2) = 2$

(d) $\frac{dy}{dx} = \left(y^2-1\right)\cos x \quad , y(0) = 2$

(e) $\frac{dy}{dt} = e^{y+t} \quad , y(0) = a$

(f) $t^2\frac{dx}{dt} = \frac{1}{x}, \quad x(1) = 9$

1ST-ORDER LINEAR DIFFERENTIAL EQUATIONS

The most general first-order linear differential equation must have the form,

$$\frac{dy}{dx} + \mathrm{p}(x)y = \mathrm{r}(x) \qquad (1)$$

where p(x) and r(x) are arbitrary functions of the independent variable here that is x.

To solve the linear differential equation (1), we need to find:

1) *the* ***complementary function,*** *i.e the general solution* y_1 *of the equivalent homogeneous equation* (2):

$$\frac{dy_1}{dx} + \mathrm{p}(x)y_1 = 0 \qquad (2)$$

i.e. set the right-hand side of equation (1) to zero.

Note: *for a system, it is called the transient solution or response.*

2) *a* ***particular integral*** y_2 *of the inhomogeneous equation* (1). A particular integral must satisfy to the differential equation (1) and can be found using
   - a *trial function* or,
   - either *the variation of the constants or the integrating factor*.

Note: for a system, it is called the steady state solution or response.

3) Finally, the **general solution** of the linear differential equation (1) is the sum of the complementary function y_1 and the particular integral y_2:

$$y = y_1 + y_2$$

4) If the *initial conditions are given* then a **particular solution** of the differential equation (1) can be found by solving the *initial value problem*.

Example using trial function: (worked during lecture)

Find the complementary function of the 1st-order linear differential equation

$$y' + \sin x . y = \sin 2x - 2\sin x \qquad (1)$$

then verify that $y_2 = 2\cos x$ is a particular integral and give the general solution.

WORK EXERCISE 3

Solve the following 1st-order linear differential equations

(a) $y'\cos x + y\sin x = 1$, try a particular integral of the form: $\sin x$.

(b) $\left(1+x^2\right)y' + xy = 1 + 2x^2$, try a particular integral of the form: x.

USING THE METHOD OF VARIATION OF CONSTANTS

This is a general method for finding the particular integral from the complementary function.

Example: Solve (1): $\dfrac{dx}{dt} + tx = t$, using the variation of the constants.

1) Complementary function: general solution of the equivalent homogeneous equation (2):

$$\frac{dx}{dt} + tx = 0 \qquad (2)$$

$$\frac{dx}{dt} = -tx$$

$$\int \frac{dx}{x} = -\int tdt$$

$$\ln\left|\frac{x}{K}\right| = -\frac{t^2}{2}$$

$$x = Ke^{-\frac{t^2}{2}}$$

The general solution of (2), i.e. complementary function of (1) is $x_1 = Ke^{-\frac{t^2}{2}}$, where K is constant.

2) Particular integral of (1):

Set a particular integral so that the constant K becomes a function of x, $x_2 = K(x)e^{-\frac{t^2}{2}}$ and find $K(x)$ so that x_2 satisfies to equation (1). Thus,

$$x_2{}' = K'(x)e^{-\frac{t^2}{2}} - K(x).te^{-\frac{t^2}{2}}$$

Substituting in (1) gives,

$$K'(x)e^{-\frac{t^2}{2}} - K(x).te^{-\frac{t^2}{2}} + tK(x)e^{-\frac{t^2}{2}} = t$$

$$K'(x)e^{-\frac{t^2}{2}} = t$$

$$K(x) = \int te^{\frac{t^2}{2}}dx + Const$$

$$K(x) = e^{\frac{t^2}{2}} + Const$$

Set $Const = 0$, thus the particular integral of (1) is

$$x_2 = K(x)e^{-\frac{t^2}{2}}$$

$$x_2 = e^{\frac{t^2}{2}}.e^{-\frac{t^2}{2}}$$

$$x_2 = 1$$

Finally, the general solution of (1) is $x = Ke^{-\frac{t^2}{2}} + 1$.

WORK EXERCISE 4

Find the general solution of the following differential equations using the variations of the constants then solve the initial value problem and give the corresponding particular solution:

(a) $\dfrac{dy}{dt} + 3y = t, \quad y(0) = 1$

(b) $y' - \dfrac{y}{x} = x^2 - 3, \quad y(1) = -1$

(c) $y' - 2t(2y-1) = 0, \quad y(0) = 0$

(d) $\dfrac{dx}{dt} + 5x - t = e^{-2t}, \quad x(-1) = 0$

(e) $\dfrac{dy}{dy} + (y-U)\sin t = 0, \quad y(\pi) = 2U$

USING THE INTEGRATING FACTOR

The integrating factor method can be applied to equations of the form:

$$\frac{dy}{dx} + \mathrm{p}(x)y = \mathrm{r}(x) \qquad (1)$$

This is actually a shorter version of the variation of constants.

The integrating factor for:

$$\frac{dy}{dx} + \mathrm{p}(x)y = \mathrm{r}(x) \qquad (1)$$

is given by

$$\mu(x) = e^{\int \mathrm{p}(x)dx}$$

and the solution of the equation is obtained from

$$\mu(x)y = \int \mu(x)r(x)dx$$

Example: Solve (1): $\frac{dx}{dt}+tx=t$, using the using the integrating factor method.

Referring to the standard first order linear equation, we identify the dependent variable, *x*, and the independent variable *t*, and

$$\frac{dx}{dt}+p(t)x=r(t)$$

where $p(t)=t$ and $r(t)=t$. Hence, the integrating factor is defined by

$$\mu(t)=e^{\int tdt}$$
$$=e^{\frac{t^2}{2}}$$

and the general solution result from

$$\mu(t)x=\int \mu(t)r(t)dt \qquad (2)$$

Differentiate (2) gives:

$$\frac{d}{dt}[\mu(t)x]=\mu(t)r(t)$$
$$\frac{d}{dt}\left[e^{\frac{t^2}{2}}.x\right]=e^{\frac{t^2}{2}}.t$$
$$e^{\frac{t^2}{2}}.x=\int te^{\frac{t^2}{2}}dt$$
$$e^{\frac{t^2}{2}}.x=e^{\frac{t^2}{2}}+C$$

Thus, the general solution is,

$$x=1+Ce^{-\frac{t^2}{2}}$$

WORK EXERCISE 5

Same as work exercise 4, but using the integrating factor method.

Solving 2nd-order differential equations

2ND-ORDER LINEAR CONSTANT COEFFICIENT DIFFERENTIAL EQUATIONS

The most general form a 2nd-order constant coefficient differential equation can take, is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad (1)$$

where *a*, *b* and *c* are constant coefficients.

To solve the differential equation (1), we need to:

1) *Find the **complementary function**: general solution y_1 of the equivalent homogeneous equation*:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \qquad (2)$$

2) *Find a **particular integral** y_2 of the inhomogeneous equation* (1):
   a) *By trial based on the nature of* $f(x)$ (polynomial, sinusoid, exponential…)
   b) *Use either the variation of the constants or the integrating factor.*
3) *Finally, the **general solution** of the differential equation (1) is the sum of the complementary function y_1 and the particular integral y_2:*

$$y=y_1+y_2$$

4) *If the initial conditions are given, give the **particular solution** by solving the initial value problem*

Complementary function: general solution of the equivalent homogeneous equation:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \qquad (2)$$

The general solution of the equation (2) depends on the values of *a*, *b* and *c*. Thus, we need to extract and solve the characteristic equation.

Setting $m\equiv\frac{dy}{dx}$, yield the characteristic equation,

$$am^2+bm+c=0 \qquad (3)$$

The roots of the characteristic equation depends on the discriminant:

$$\Delta=b^2-4ac$$

- If $\Delta > 0$ $(b^2 > 4ac)$ the characteristic equation has two real and distinct roots,

$$\begin{matrix} m_1 \\ m_2 \end{matrix} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

and *the general solution of the equivalent homogeneous equation* (2) (the complementary function of (1)) *is*,

$$y_1 = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

where C_1, C_2 are constant.

- If $\Delta = 0$ $(b^2 = 4ac)$ the characteristic equation has two real and identical roots,

$$m_1 = m_2 = m = \frac{-b}{2a}$$

then *the general solution of the equivalent homogeneous equation* (2) *is*,

$$y_1 = [C_1 + C_2 t]e^{mt}$$

where C_1, C_2 are constant.

- If $\Delta < 0$ $(b^2 < 4ac)$ the characteristic equation has a pair of complex conjugate roots,

$$\begin{matrix} m_1 \\ m_2 \end{matrix} = m \pm j\omega = \frac{-b \pm j\sqrt{-\Delta}}{2a}$$

then *the general solution of the equivalent homogeneous equation* (2) *is*,

$$y_1 = e^{mt}[C_1 \cos \omega t + C_2 \sin \omega t]$$

where C_1, C_2 are constant.

Example: Solve the homogeneous equations (worked during lecture)

(a) $\frac{d^2 y}{dx} - 9\frac{dy}{dx} + 6y = 0$

(b) $y'' + 6y' + 9y = 0$

WORK EXERCISE 5

Find the general solution of the following homogeneous differential equations:

(a) $y''(x) - 3y'(x) + 2y(x) = 0$

(b) $\ddot{y} + 4\dot{y} + 4y = 0$ (c) $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0$

(d) $y'' - 4y = 0$ (e) $\frac{d^2 y}{dt^2} + \frac{dy}{dt} = 0$

Finding a particular integral of the inhomogeneous equation

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad (1)$$

We have found the general solution of the equivalent homogeneous equation (2) depending on the roots of the characteristic equation (3).

The trial function chosen as particular integral depends on the nature of $f(x)$, i.e. the nature of the right-hand side of the differential equation.

The particular integral is found by assuming the general form of the function $f(x)$, substituting this in the differential equation and equating the coefficients. An example will make this clear.

Example: Find a particular integral of

$$\frac{d^2 y}{dt^2} - 5\frac{dy}{dt} + 6y = x^2 \qquad (1)$$

More generally, if the right-hand side $f(x)$ *has the following form then assume the corresponding particular integral*:

$f(x) = k$	$\rightarrow$	$y = C$
$f(x) = kx$	$\rightarrow$	$y = Cx + D$
$f(x) = kx^2$	$\rightarrow$	$y = Cx^2 + Dx + E$
$f(x) = k \sin x$ or $k \cos x$	$\rightarrow$	$y = C\cos x + D\sin x$
$f(x) = k \sin(nx)$ or $k \cos(nx)$	$\rightarrow$	$y = C\cos(nx) + D\sin(nx)$
$f(x) = e^{kx}$	$\rightarrow$	$y = Ce^{kx}$

WORK EXERCISE 6

Suppose the following functions $f(x)$ are the right-hand side of differential equations. Give the corresponding particular integral that should be defined to solve the differential equations if they were given.

(a) $f(x) = 2x - 3$ (b) $f(x) = e^{5x}$

(c) $f(x) = 3 - 5x^2$ (d) $f(x) = \sin 4x$

(e) $f(x) = 27$ (f) $f(x) = 5\cosh 4x$