Introduction to the Discrete-Time Fourier Transform and the DFT

C.S. Ramalingam

Department of Electrical Engineering
IIT Madras

The Discrete-Time Fourier Transform

- The DTFT tells us what frequency components are present

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- $|X(\omega)|$: magnitude spectrum
  $\angle X(\omega)$: phase spectrum

- E.g.: $\exp(j\omega_0 n)$ has only one frequency component at $\omega = \omega_0$
  - $\exp(j\omega_0 n)$ is an infinite duration complex sinusoid
  - $X(\omega) = 2\pi\,\delta(\omega - \omega_0) \qquad \omega \in [-\pi, \pi)$
  - the spectrum is zero for $\omega \neq \omega_0$

- $\cos(\omega_0 n)$ and $\sin(\omega_0 n)$ have frequency components at $\pm\omega_0$
  - phase spectra for sin and cos are different

The Discrete-Time Fourier Transform

- The DTFT is periodic with period 2π

$$\begin{aligned} X(\omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi)n} \\ &= X(\omega) \end{aligned}$$

- $X(\omega)$ is also commonly denoted by $X(e^{j\omega})$
  - the notation $X(e^{j\omega})$ conveys the periodicity explicitly
- $X(\omega)$ over one period contains all the information
  - typically we consider either $[0, 2\pi)$ or $[-\pi, \pi)$
- DTFT of $\exp(j\omega_0 n)$ over *all* ω:

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 + 2\pi k)$$

Inverse DTFT

- $x[n] = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$

- Example:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\, \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega_0 n}$$

- If $\omega_0 = 0$, then $x[n] = 1$ for all n, i.e., DC sequence
  Its transform is an impulse located at $\omega = 0$ with strength 2π

Convolution-Multiplication Property

- Multiplication in one domain is equivalent to convolution in the other domain
  - $x[n] \cdot y[n] \overset{\text{DTFT}}{\longleftrightarrow} \dfrac{1}{2\pi} X(\omega) \circledast Y(\omega)$
  - $x[n] * y[n] \overset{\text{DTFT}}{\longleftrightarrow} X(\omega) \cdot Y(\omega)$

- $x[n] * y[n] = \displaystyle\sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$

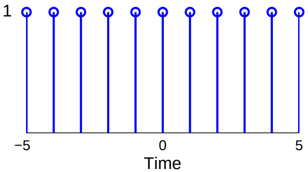
$$X(\omega) \circledast Y(\omega) = \int_{-\pi}^{\pi} X(\theta)\, Y(\omega - \theta)\, d\theta$$

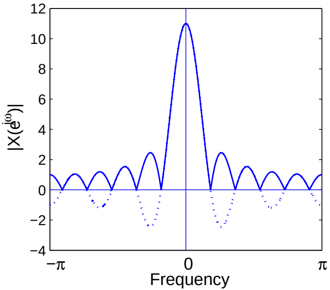
Rectangular Window and its Transform

- Rectangular window:

$$w[n] = 1 \qquad n = -N, \ldots, 0, \ldots N$$

- $W(\omega) = \dfrac{\sin(2N+1)\omega/2}{\sin \omega/2}$





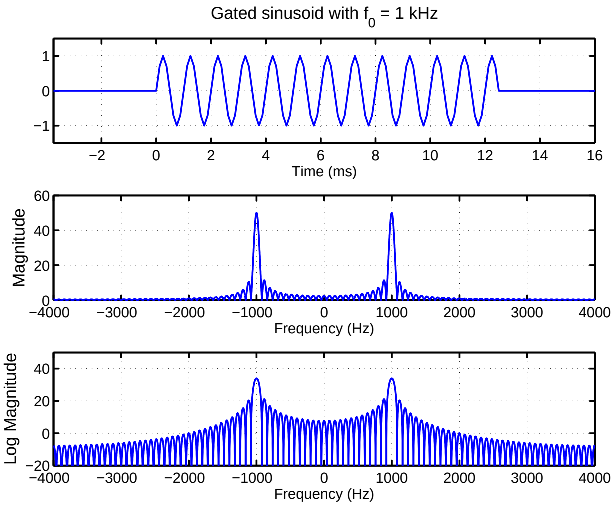
Some Observations

- $W(e^{j\omega})\Big|_{\omega=0} = 2N+1$
- First zero crossing occurs when $\omega = \dfrac{2\pi}{2N+1}$
- Number of zero crossings $= 2N$
- As N increases, main lobe height increases and width decreases
- Transform of $\exp(j\omega_0 n)\, w[n]$ has its mainlobe centred at ω_0

$$\begin{aligned}\text{DTFT}\left(x[n]\cdot w[n]\right) &= \frac{1}{2\pi} X(\omega) \circledast W(\omega) \\ &= \delta(\omega-\omega_0) \circledast W(\omega) \\ &= W(\omega-\omega_0)\end{aligned}$$

- $\text{DTFT}\left(\cos(\omega_0 n)\, w[n]\right) = \frac{1}{2} W(\omega-\omega_0) + \frac{1}{2} W(\omega+\omega_0)$

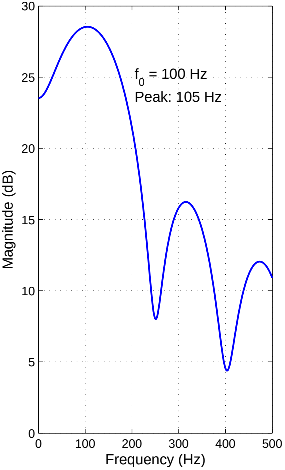
Windowed Sinusoid Example

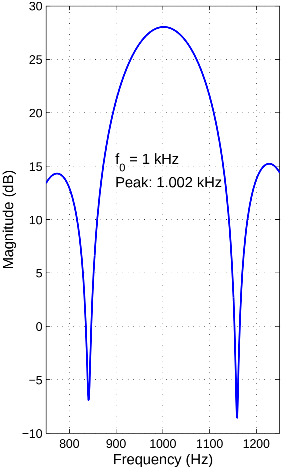


Single Real Sinusoid Frequency Estimation

- f_0 is estimated from peak location
- In general, the peaks are not exactly at $\pm f_0$
- This is because of sidelobe interference
- If f_0 is closer to DC
  - sidelobe interference increases
  - shifts peak further away from true location
- Interference is least when sinusoidal frequency is at $f_s/4$

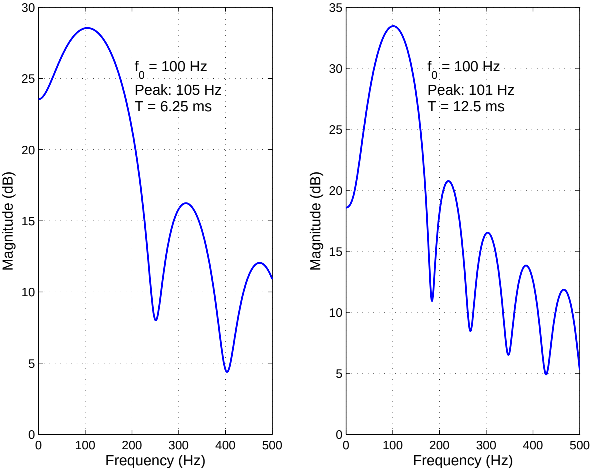
Examples of Peak Shifting





Effect of Increased Data Length

- Longer duration sinusoid's spectrum is narrower
  - less sidelobe interference ⇒ peak closer to true value



Use of Data Windows

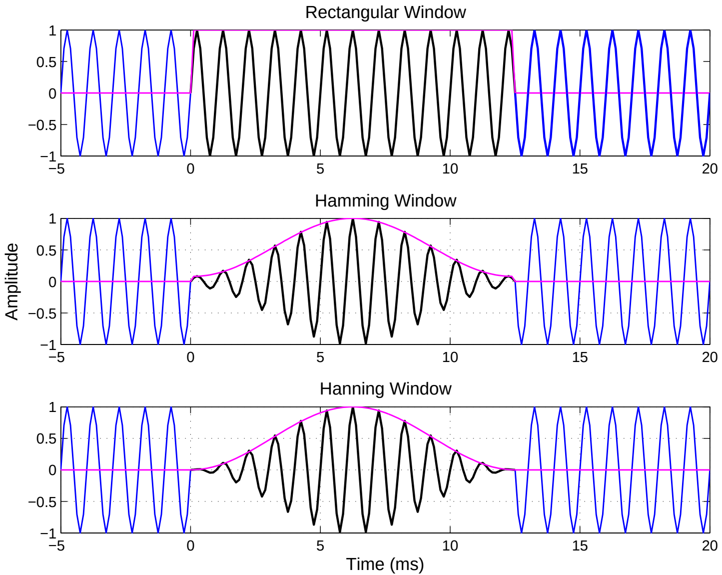
- Useful for data containing sinusoids
- Sidelobes of a stronger sinusoid will mask the main lobe of a nearby weak sinusoid
- We multiply $x[n]$ by data window $w[n]$ before computing the DTFT
  - if we merely truncate a signal, it is equivalent to applying a rectangular window
- Why consider non-rectangular windows?
  - sidelobes fall of faster
  - nearby weaker sinusoid becomes more visible
  - price paid: main lobe of each sinusoid broadens
    - two close peaks may merge into one

Commonly Used Windows

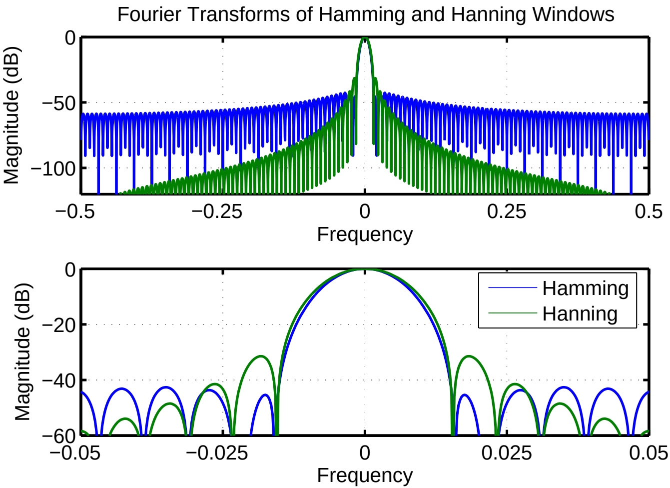
Name	$w[k]$	Fourier transform
Rectangular	1	$W_R(f) = \dfrac{\sin\ \pi f(2N+1)}{\sin\ \pi f}$
Bartlett	$1 - \dfrac{\lvert k\rvert}{N}$	$\dfrac{1}{N}\left(\dfrac{\sin\ \pi f N}{\sin\ \pi f}\right)^2$
Hanning	$0.5 + 0.5\cos\dfrac{\pi k}{N}$	$0.25\, W_R\left(f - \frac{1}{2N}\right) + 0.5\, W_R(f) +$ $0.25\, W_R\left(f + \frac{1}{2N}\right)$
Hamming	$0.54 + 0.46\cos\dfrac{\pi k}{N}$	$0.23\, W_R\left(f - \frac{1}{2N}\right) + 0.54\, W_R(f) +$ $0.23\, W_R\left(f + \frac{1}{2N}\right)$

$w[k] = 0$ for $\lvert k\rvert > N$

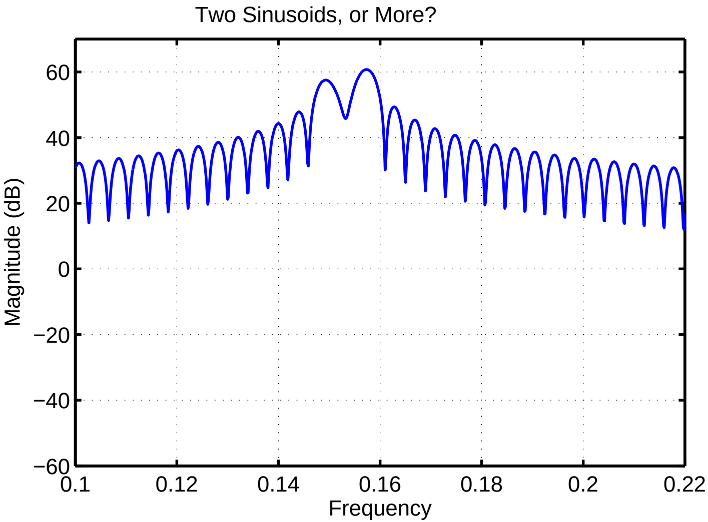
Windowed Sinusoid



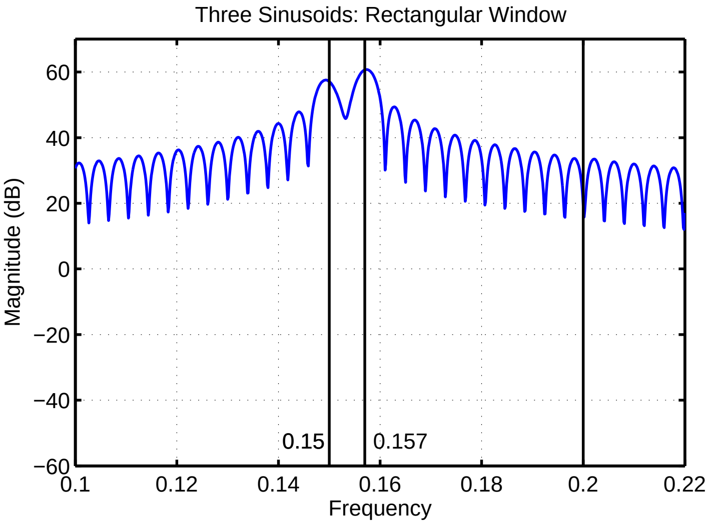
Hamming Vs. Hanning



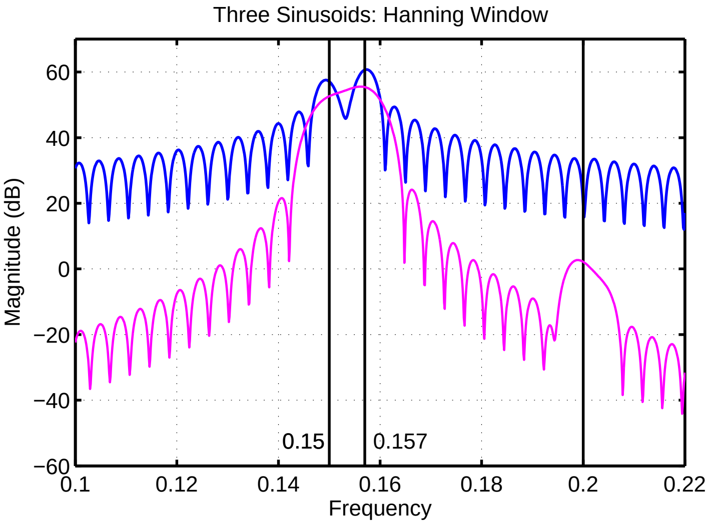
Example: How Many Sine Waves Are there?



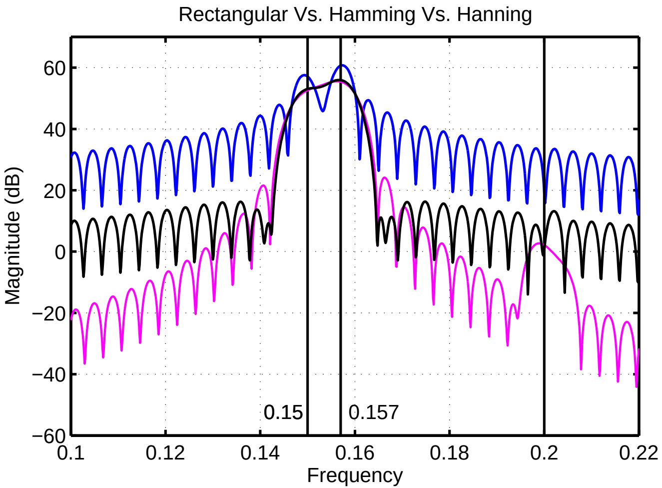
Example: Three Sine Waves



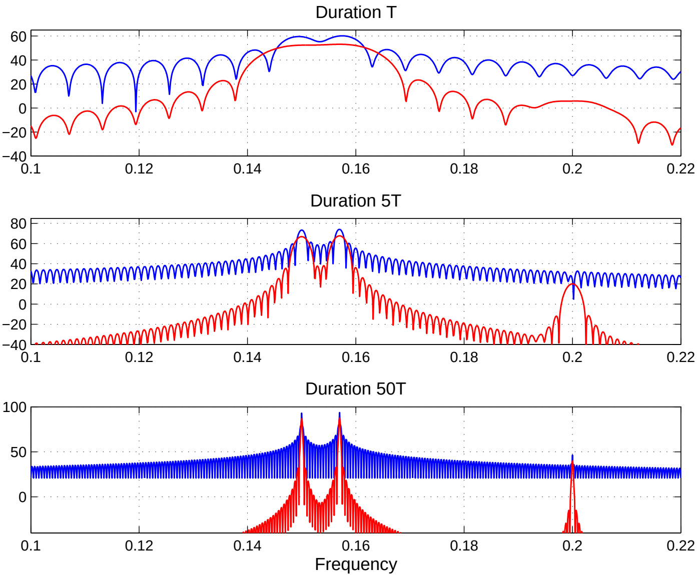
Example: Three Sine Waves



Three Sine Waves



Three Sine Waves: Three Diffrent Data Lengths



The Discrete Fourier Transform

- Since ω is a continuous variable, $X(\omega)$ cannote be evaluated on a computer
- The Discrete Fourier Transform (DFT) is amenable to machine compuation
- Let $x[n]$ be defined over the interval $0, 1, \ldots, N-1$ and zero otherwise
- $$\boxed{X[k] \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N} n} \qquad k = 0, 1, \ldots, N-1}$$
  - $X[k+N] = X[k]$ i.e., only N distinct values are present
  - The $X[k]$'s are called the DFT coefficients

- The inversion formula is

$$\boxed{\tilde{x}}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k}{N} n}$$

Why is the inverse $\tilde{x}[n]$ and not $x[n]$?

- $\tilde{x}[n+N] = \tilde{x}[n]$, i.e., inverse is periodic with period N

  $x[n] = \tilde{x}[n]$ for $n = 0, 1, \ldots, N-1$
- Even though we start off with an *aperiodic* signal, the inverse transform gives a *periodic* signal
- But over the fundamental period, the inverse transform equals the original aperiodic signal

DFT = Sampled Version of DTFT

- Recall

$$X(\omega) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

- Evaluate $X(\omega)$ at N uniformly spaced points in the interval $[0, 2\pi)$, i.e.,

$$\begin{aligned} X(\omega)\big|_{\omega=\frac{2\pi k}{N}} &= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n} \\ &= X[k] \end{aligned}$$

- DFT coefficients can be viewed as samples of $X(\omega)$
- Since $X(\omega + 2\pi) = X(\omega)$, the samples of $X(\omega)$ are also periodic
  - provides another explanation for why $X[k+N] = X[k]$

Frequency domain sampling introduces time-domain periodicity!

- Sampling in the frequency domain leads to periodic repetition in the time domain
- Repetition period is N
- If we sample the DTFT at L $(> N)$ points, the repetition period will be L $(> N)$
- If $x[n]$ is of duration N, then $X(\omega)$ has to be sampled at least at N points to avoid aliasing in the time domain

Effect of Zero-Padding

- $X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$

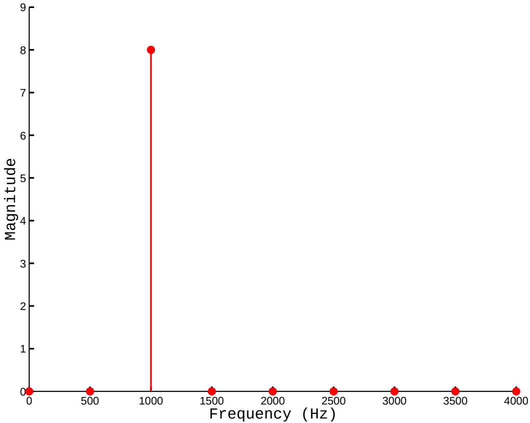
- Append $L - N$ zeros $x[n]$ and compute the L-point DFT of the padded sequence

- This is equivalent to sampling $X(\omega)$ at L $(> N)$ points:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/L} \qquad k = 0, 1, \ldots, L-1$$

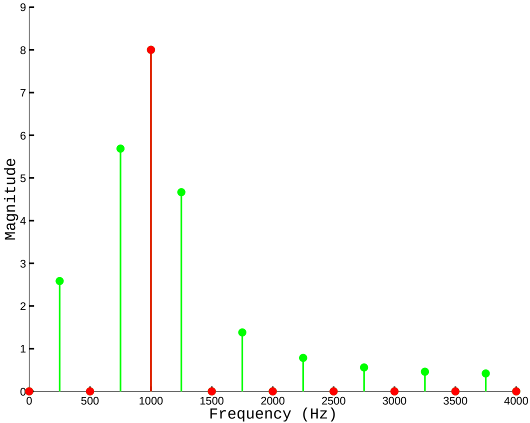
- The underlying $X(\omega)$ remains the same, since it depends only on $x[n]$, $n = 0, 1, \ldots, N-1$

Example



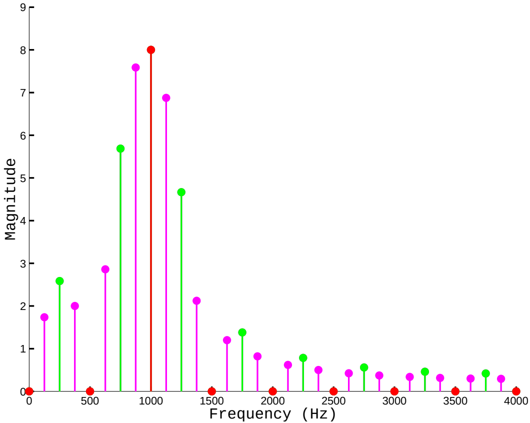
16-pt DFT of $x[n] = \sin(2\pi n/8) \qquad n = 0, 1, \ldots, 15$

Example



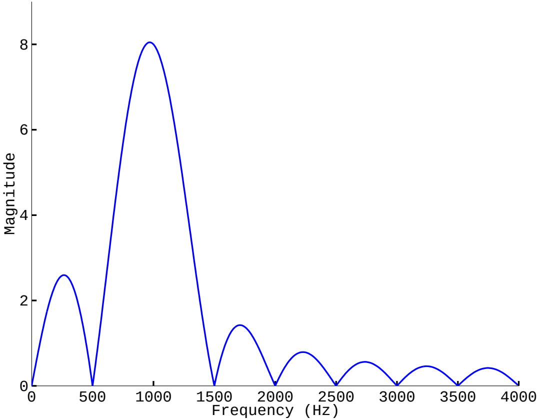
32-pt DFT of $x[n]$: 16 signal samples padded with 16 zeros

Example



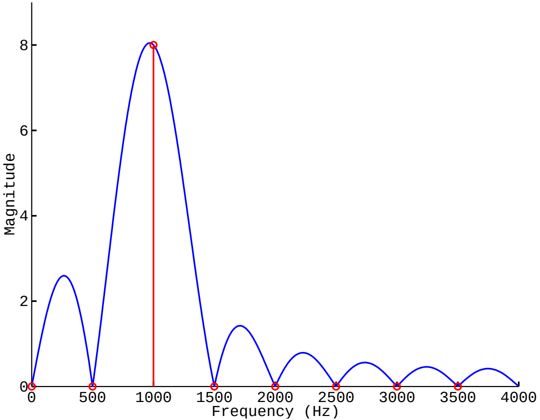
64-pt DFT of $x[n]$ (zero-padded)

Example



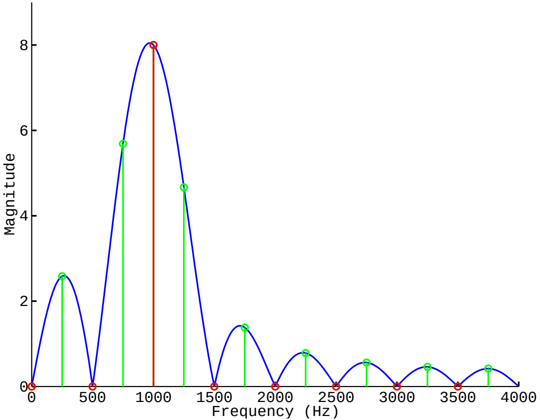
DTFT of $x[n] = \sin(2\pi n/8)$

Example



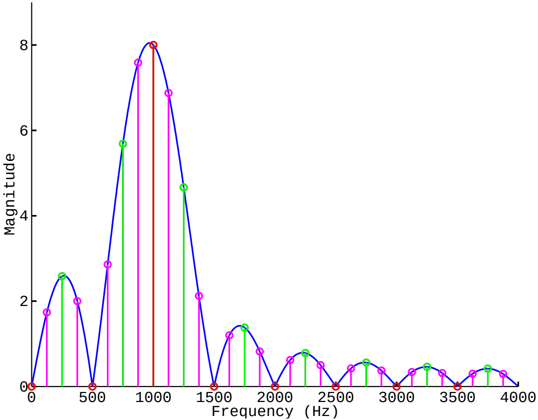
DTFT and 16-pt DFT

Example



DTFT and 32-pt DFT

Example



DTFT and 64-pt DFT

Relationship Between Analog and Digital Spectra

- Recall $x[n] = x(nT_s)$
- $x(t) \xleftrightarrow{\text{CTFT}} X(\Omega)$
- $x[n]$'s DTFT $X(\omega)$ is related to $x(t)$'s CTFT $X(\Omega)$ as follows:
  - Amplitude scaling by $\frac{1}{T_s}$
  - Periodic repetition due to sampling
  - Frequency axis scaling by $F_s = \frac{1}{T_s}$

Relationship Between Analog and Digital Frequencies

- A frequency ω_0 (f_0) in the DTFT corresponds to $\omega_0 \cdot F_s$ rad/s ($f_0 \cdot F_s$ Hz)

- Converting DFT bin to digital and analog frquencies:

  Let $X[k]$ be an N-point DFT. The digital and analog frequencies corresponding to bin k are:

  - 0-based index: $\dfrac{k}{N}$ $\qquad$ $\dfrac{k}{N} \cdot F_s$ Hz
  - 1-based index: $\dfrac{k-1}{N}$ $\qquad$ $\dfrac{k-1}{N} \cdot F_s$ Hz

- If F_s is not known, it is not possible to know the true analog frequency given knowledge about DTFT/DFT