Problem 2.1

Evaluate the Fourier transform of the damped sinusoidal wave $g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$ where $u(t)$ is the unit step function

Solution

The Fourier transform of $g(t)$ is

$$
\begin{aligned}
G(f) &= \int_0^\infty \exp(-t)\sin(2\pi f_c t)\sin(-j2\pi f_c t)dt \\
&= \frac{1}{2j}\int_0^\infty \exp(-t)[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]\exp(-j2\pi f t)dt \\
&= \frac{1}{2j}\int_0^\infty [\exp(j2\pi(f_c - f)t - t)]dt \\
&= \frac{1}{2j}\left[\frac{1}{j2\pi(f_c - f) - 1}\exp(j2\pi(f_c - f)t - t) + \frac{1}{j2\pi(f_c - f) + 1}\exp((-j2\pi(f_c + f)t) - t)\right]_{t=0}^{\infty} \\
&= \frac{1}{2j}\left(\frac{1}{j2\pi(f_c - f) - 1} + \frac{1}{j2\pi(f_c - f) + 1}\right) \\
&= \frac{1}{2j}\left(\frac{(j2\pi(f_c - f) + 1) + (j2\pi(f_c - f) - 1)}{1 + 4\pi^2(f_c - f)^2}\right) \\
&= \frac{2\pi f_c}{1 + 4\pi^2(f - f_c)^2}
\end{aligned}
$$