Linear Differential Equations

1.0 Introduction

Invariably any accurate model of a dynamic system will require the use of differential equations. In this course will we look at first order and second order systems, initially in terms of the ‘mathematical’ solution of their equations but developing an ‘engineering’ method that allows the solution of ordinary differential equations of any order. This method has another advantage in that it removes the need for integration or differentiation.

2.0 First Order Differential Equations

First order dynamic systems contain a single time varying element, two examples of first order systems are shown in Figure 2.1. For the electrical circuit the relationship between the applied voltage, V and the circulating current, i is:

$$V = iR + L\frac{di}{dt} \quad \text{or} \quad \frac{V}{R} = i + \frac{L}{R}\frac{di}{dt} \qquad \{2.1\}$$

For the mechanical circuit the following holds true:

$$F = Bv + M\frac{dv}{dt} \quad \text{or} \quad \frac{F}{B} = v + \frac{M}{B}\frac{dv}{dt} \qquad \{2.2\}$$

The solution of these equations consists of a Complimentary Function (CF) and a Particular Integral (PI). For the electrical circuit the overall solution is the current flowing in the circuit after a predefined time. The CF represents the response of the circuit to its initial conditions and the PI is the response due to the input. For the electrical circuit it is necessary to consider two possible situations that may occur to demonstrate the influence of the initial conditions. Firstly, consider the following set of conditions:

$$\left.\begin{array}{ll} t < 0: & i = \dfrac{V}{R} \\ t = 0 & i_0 = \dfrac{V}{R}, \quad V = 0 \\ t > 0 & i = ? \end{array}\right\} \qquad \{2.3\}$$

Under these conditions the response of the circuit will be a decay of the initial current through the inductor (Figure 2.1). However, if the conditions are altered the response of the circuit will be different. Consider

$$\left.\begin{array}{ll} t < 0: & i = 0, \quad V = 0 \\ t = 0 & V = V_0 \\ t > 0 & i = ? \end{array}\right\} \qquad \{2.4\}$$

In this case the current will increase until it reaches a maximum value of V_0/R (Figure 2.1).

2.1 Response to initial conditions: The Complimentary Function

To determine the response to initial conditions it is necessary to assume that there is no input into the system. In the case of the electrical circuit this means that:

$$0 = i + \frac{L}{R}\frac{di}{dt} \qquad \{2.5\}$$

To solve this equation a function is required that when differentiated gives the same function only negative. Therefore, let the current be defined as

$$i = \alpha e^{\beta t} \qquad \{2.6\}$$

Differentiating the current gives

$$\frac{di}{dt} = \alpha\beta e^{\beta t} \qquad \{2.7\}$$

Substituting Equations 2.6 and 2.7 into Equation 2.5 gives

$$\left(\frac{L}{R}\beta + 1\right)\alpha e^{\beta t} = 0 \qquad \{2.8\}$$

For a non-trivial solution

$$\beta = -\frac{R}{L} \qquad \{2.9\}$$

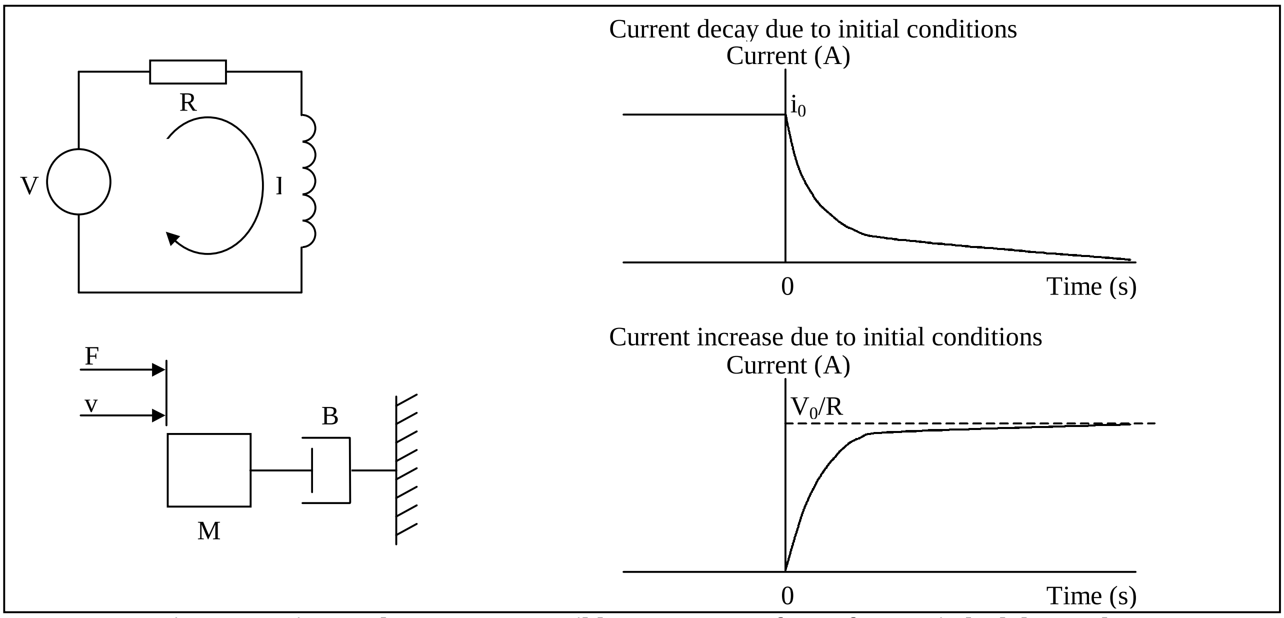


Figure 2.1 First Order Systems; Possible current waveforms for a switched dc supply

When t is zero the initial current flowing (from Equations 2.3 and 2.6) is

$$i_0 = \frac{V}{R} = \alpha e^0 = \alpha \qquad \{2.10\}$$

Therefore α is the initial current, i_0, which is equal to V/R in this case. If the current is plotted with respect to time (Figure 2.2) then the gradient of the initial slope can be determined. The shape of the current waveform is dependent on L and R, if the initial slope is very steep then the circuit will respond rapidly to fluctuations in the supply voltage. On the other hand a circuit that generates a shallow gradient will be far more sluggish. Rather than use the gradient of the initial slope as a comparative measure of system response the *Time Constant* (T) is specified. If the system response was just the initial slope rather than an exponential decay then the Time Constant is defined as the time taken for the initial value to decay to zero (See Figure 4.2). For systems were the initial slope is positive e.g. the current is increasing rather than decaying, then the Time Constant can be defined as the time taken for the response to reach 63% of its final value. Another useful rule of thumb is that a first order system will be within 98% of its steady state (or final) value within 4T seconds

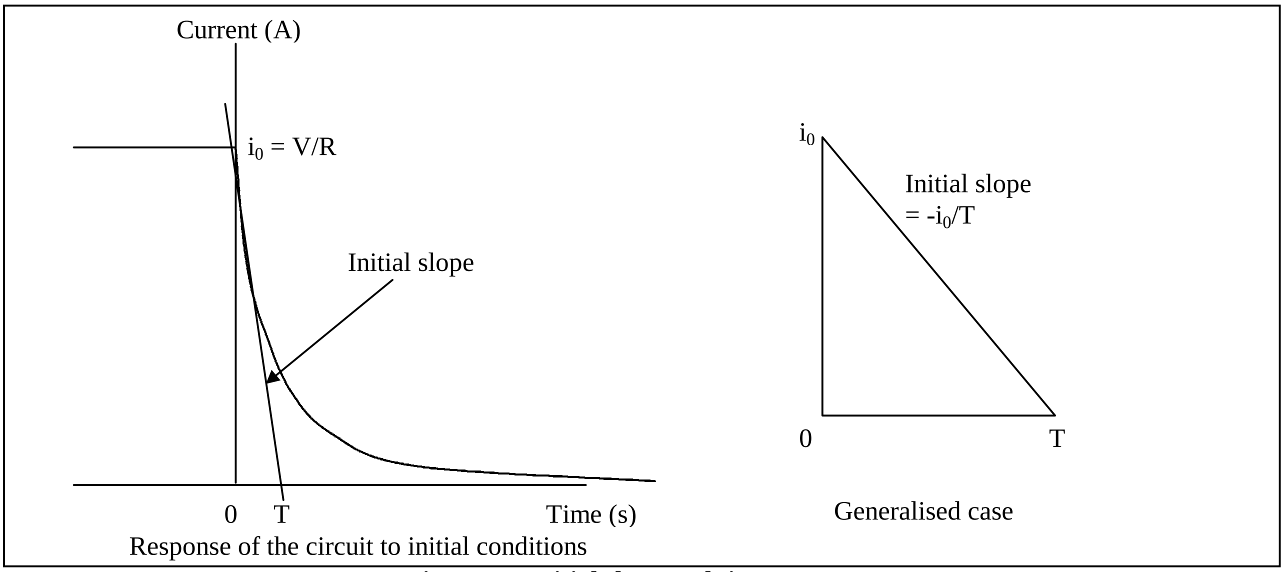


Figure 2.2 Initial slope and time constant

Substituting the values for α and β into Equation 2.6 gives

$$i = \frac{V}{R} e^{-\frac{Rt}{L}} \qquad \{2.11\}$$

Therefore evaluating di/dt when t is equal to zero will give the initial slope, which can be compared with the generalised case shown in Figure 2.2. That is

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{V}{R}\frac{R}{L} = \frac{-i_0}{T} \qquad \{2.12\}$$

Therefore β is equal to -1/T. Given that by analogous components we know that all first order systems will have identical equations (with different variables) we can convert the solution of the complimentary function into a generalised case:

$$\begin{aligned} &V = iR + L\frac{di}{dt} \quad \text{or} \quad \frac{V}{R} = i + \frac{L}{R}\frac{di}{dt} \quad \text{the CF is} \quad i = \frac{V}{R}e^{-\frac{R}{L}t} \\ &\therefore \quad \text{if} \quad y = x + T\frac{dx}{dt} \quad \text{and } x = x_0, t = 0 \quad \text{the CF is} \quad x = x_0 e^{-\frac{t}{T}} \end{aligned} \qquad \{2.13\}$$

2.2 Response to an input: The Particular Integral

To determine the system response to an input the initial conditions are ignored. Because the system is linear then it is possible to apply superposition. This means that the effects of initial conditions and the input can be determined separately and then summed together to provide an overall response.

2.2.1 A Step Input

Consider the case where the RL circuit of Figure 2.1 experiences a step change in applied voltage at time t = 0 such that

$$\begin{aligned} V &= 0 \text{ for } t < 0 \\ V &= V_0 \text{ for } t \geq 0 \end{aligned} \qquad \{2.14\}$$

This input can be written as (see Figure 2.3)

$$V = V_0 U(t) \qquad \{2.15\}$$

Where U(t) is defined as the unit step function

$$\begin{aligned} U(t) &= 0 \text{ for } t < 0 \\ U(t) &= 1 \text{ for } t \geq 0 \end{aligned} \qquad \{2.16\}$$

From knowledge of circuit theory it is known that the current flowing after 0 seconds will increase until it reaches a steady value. The *steady state* can be defined as the value of the current after infinite time, before the steady state is achieved there is a *transient state* during which the current will vary with respect to time. Therefore a system has two responses to an input; the transient response and the steady state response. Both are important, invariably the design engineer is interested in the steady state response of a system, however poor design can lead to systems that satisfy steady state requirements but whose transient response is very oscillatory leading to poor dynamic performance and ultimately instability. The current flowing due to initial conditions (Equation 2.11) is not a suitable solution in this case because it will be zero as time tends to infinity, that is

$$\lim_{t\to\infty} \quad \frac{V}{R}e^{-\frac{R}{L}t} \to 0 \qquad \{2.17\}$$

For the RL circuit it is known that

$$\lim_{t\to\infty} \quad i \to \frac{V_0}{R} \qquad \{2.18\}$$

Therefore, using some intuition, a reasonable guesstimate of the current flowing due to a step change voltage input is

$$i(t) = \delta + \varepsilon e^{-\frac{R}{L}t} \qquad t \geq 0 \qquad \{2.19\}$$

The two unknown constants can be determined by calculating the current flowing at 0 seconds and after infinite time.

$$\left.\begin{aligned} &t = 0, \quad i = 0, e^0 = 1, \quad \therefore \varepsilon = -\delta \\ &t = \infty, \quad i = \frac{V_0}{R}, e^{-\infty} = 0 \quad \therefore \delta = \frac{V_0}{R} \end{aligned}\right\} \qquad \{2.20\}$$

Therefore the current flowing after the step change voltage has been applied is

$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right) \qquad t \geq 0 \qquad \{2.21\}$$

This equation is incorrect for negative t (i.e. before the step change input was applied) as in limit if time were tending towards minus infinity then the equation would yield a current of minus infinity Amps, whereas we know that the current was zero. Therefore to be correct for all time the answer should be multiplied by the unit step function (which is zero for negative time) i.e.

$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)U(t) \qquad \{2.22\}$$

To check that this is a correct solution we can substitute the values for i(t) and di/dt back into the original equation which was

$$\frac{V}{R} = i + \frac{L}{R}\frac{di}{dt} \tag{2.1}$$

Differentiating Equation 2.22 gives

$$\frac{di}{dt} = \left(\frac{V_0}{L} e^{-\frac{R}{L}t}\right) U(t) \tag{2.23}$$

Consider the LHS of Equation 2.1, substituting for i(t) and di/dt gives

$$i + \frac{L}{R}\frac{di}{dt} = \left(\frac{V_0}{R} - \frac{V_0}{R} e^{-\frac{R}{L}t} + \frac{L}{R}\frac{V_0}{L} e^{-\frac{R}{L}t}\right) U(t) = \frac{V_0 U(t)}{R} \tag{2.24}$$

Which (given Equation 2.15) is equal to the RHS of Equation 2.1 QED. As with the complimentary function we can rewrite this result so that it is applicable to the general case, that is

$$\text{if} \quad y = x + T\frac{dx}{dt} \quad \text{and} \quad y = U(t) \quad \text{then} \quad x = k\left(1 - e^{-\frac{t}{T}}\right) U(t) \tag{2.25}$$

Where T is the system time constant and k is the steady state gain.

The overall response of the circuit is defined as the complimentary function plus the particular integral, therefore the total current flowing after a step input has been applied to the circuit shown in Figure 2.1 is

$$i(t) = i_0 e^{-\frac{t}{T}} + \frac{V_0}{R}\left(1 - e^{-\frac{t}{T}}\right) U(t) \quad \text{where} \quad T = \frac{L}{R} \tag{2.26}$$

2.2.2 A Ramp Input

A ramp input has a constant rate of change (see Figure 2.3). If the circuit shown in Figure 2.1 where to experience a ramp change in voltage then the current would respond by rising slowly at first but then at an increasing rate. This is different to the response to a step input where the current attained a final steady state value. Therefore the solution must contain a time varying term. A ramp input is defined as

$$V = ktU(t) \tag{2.27}$$

Where k is the ramp constant. If k is unity then it is a unit ramp input. A reasonable guesstimate of the response of the circuit would be to assume that it would have similar features to the step response and an additional time varying term, that is

$$i(t) = \left(\varphi t + \gamma e^{-\frac{R}{L}t} + \sigma\right) U(t) \tag{2.28}$$

Consider the case at time t = 0

$$t = 0, \quad i = 0, \quad \therefore 0 = \gamma + \sigma, \quad \sigma = -\gamma \tag{2.29}$$

It is pointless to consider the case at infinite time as the current does not attain a final steady state value. The easiest way to determine if the current response is correct is to consider whether the response satisfies the differential equation.

Differentiating i(t) gives

$$\frac{di}{dt} = \left(\varphi - \gamma\frac{R}{L} e^{-\frac{R}{L}t}\right) U(t) \tag{2.30}$$

Substituting Equations 2.27, 2.28 and 2.30 into Equation 2.1gives

$$\frac{ktU(t)}{R} = \left\{\varphi t + \gamma e^{-\frac{R}{L}t} - \gamma + \frac{L}{R}\left(\varphi - \gamma\frac{R}{L} e^{-\frac{R}{L}t}\right)\right\} U(t) \tag{2.31}$$

This simplifies to

$$\frac{kt}{R} = \varphi t + \frac{L}{R}\varphi - \gamma \tag{2.32}$$

Equating terms containing ‘t’ gives

$$\frac{k}{R} = \varphi \tag{2.33}$$

and terms that are time invariant gives

$$0 = \frac{L}{R}\varphi - \gamma, \quad \Rightarrow \quad \gamma = \frac{kL}{R^2} \tag{2.34}$$

Therefore the response of the system to a ramp input is

$$i(t) = \left(\frac{kt}{R} + \frac{kL}{R^2} e^{-\frac{R}{L}t} - \frac{kL}{R^2} \right) U(t) \qquad \{2.35\}$$

Where k is the input ramp gradient

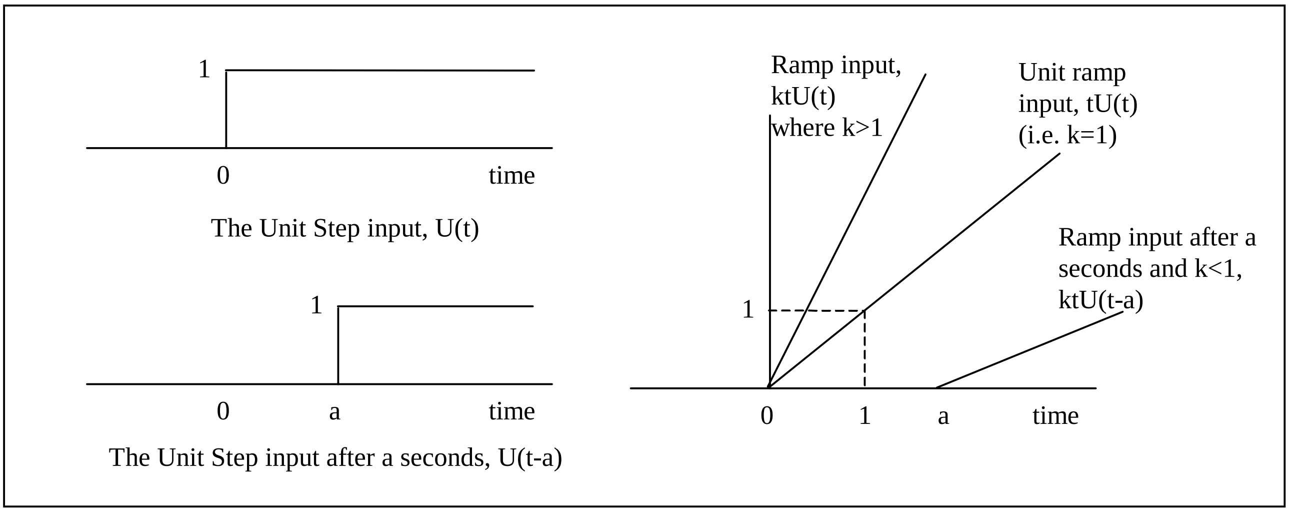


Figure 2.3 Standard input waveforms

Even with relatively simple input waveforms and a simple circuit this approach is far from ideal. It requires a good understanding of the circuit and the ability to guesstimate the most likely form of the output current. Simple second order problems will now be briefly considered prior to explaining a different approach to solving differential equations.

3.0 Second Order Differential Equations

Second order dynamic systems contain two time varying elements, two examples of second order systems are shown in Figure 3.1. Analysing the electrical circuit using Kirchoff's Voltage Law (KVL) reveals that the relationship between the applied voltage, V and the circulating current, i is:

$$V = iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad \text{or as} \quad i = \frac{dq}{dt} \quad V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} \qquad \{3.1\}$$

For the mechanical circuit the following holds true:

$$F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx \qquad \{3.2\}$$

Again the solution of these equations comprises of two expressions, one evaluated to account for the initial conditions and the other to account for the response to a specified input. In order to simplify the analysis the case where a circuit consists of an ideal capacitor and an ideal inductor only will be considered. As the circuit does not contain any resistance the equation becomes

$$V = L\frac{d^2q}{dt^2} + \frac{q}{C} \quad \text{or} \quad CV = CL\frac{d^2q}{dt^2} + q \qquad \{3.3\}$$

This is the equation for simple harmonic motion.

3.1 Response to initial conditions: The Complimentary Function

The complimentary function is the solution to

$$0 = L\frac{d^2q}{dt^2} + \frac{q}{C} = CL\frac{d^2q}{dt^2} + q \qquad \{3.4\}$$

when the initial conditions are

$$t = 0,\ q = q_0,\ i = 0 \qquad \{3.5\}$$

To solve this equation a function is required that when differentiated twice gives the same function only negative. Therefore, let the charge be defined as

$$q = A\sin\omega_0 t + B\cos\omega_0 t \qquad \{3.6\}$$

Differentiating the charge to obtain current gives

$$\frac{dq}{dt} = A\omega_0\cos\omega_0 t - B\omega_0\sin\omega_0 t \qquad \{3.7\}$$

Differentiating again gives

$$\frac{d^2q}{dt^2} = -A\omega_0^{\,2}\sin\omega_0 t - B\omega_0^{\,2}\cos\omega_0 t \qquad \{3.8\}$$

Substituting Equations 3.6 and 3.8 into Equation 3.4 gives

$$A\left(1 - CL\omega_0^{\,2}\right)\sin\omega_0 t + B\left(1 - CL\omega_0^{\,2}\right)\cos\omega_0 t = 0 \qquad \{3.9\}$$

For a non-trivial solution

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad \{3.10\}$$

The unknown constants A and B can be determined from the initial conditions. When t = 0

$$\left.\begin{aligned} &q = q_0 = A.0 + B.1 \Rightarrow B = q_0 \\ &i = \frac{dq}{dt} = 0 = A\omega_0.1 - B\omega_0.0 \Rightarrow A = 0 \end{aligned}\right\} \qquad \{3.11\}$$

Therefore the complimentary function is

$$q = q_0 \cos\left(\frac{t}{\sqrt{LC}}\right) \qquad \{3.12\}$$

This equation describes the waveform shown in Figure 3.2

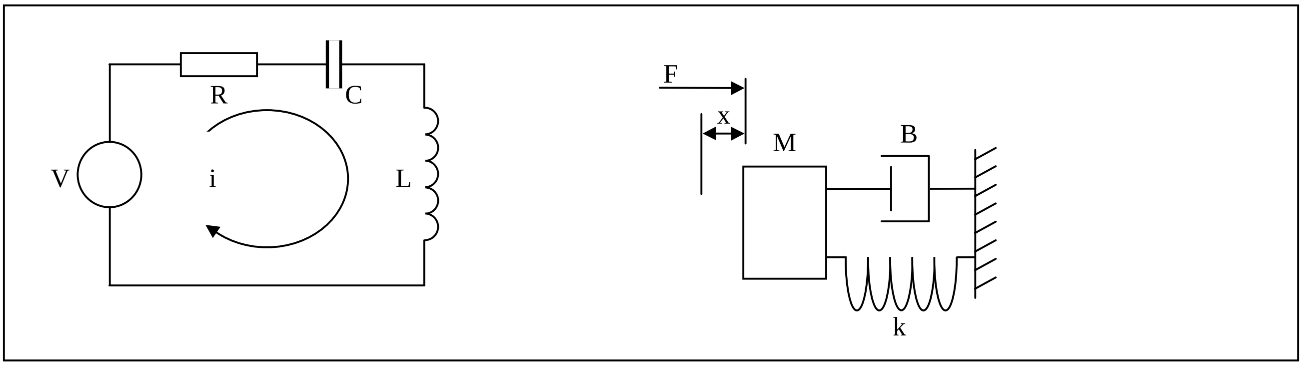


Figure 3.1 Examples of Second Order Systems

3.2 Response to an input: The Particular Integral

Consider the case when a step change in input voltage is applied to the circuit. Using the definitions of Equations 2.14, 2.15 and 2.16, the input is

$$V = V_0 U(t) \qquad \{3.13\}$$

Given that the sine term was zero due to initial conditions, it is safe to assume that only a cosine term is required. Again using the same argument as for the first order case, a good estimate of the response is to let the charge be defined as

$$q(t) = \alpha + \beta\cos\omega_0 t \qquad \{3.14\}$$

Differentiating twice gives

$$\frac{d^2q}{dt^2} = -\beta\omega_0^{\,2}\cos\omega_0 t \qquad \{3.15\}$$

Substituting Equations 3.13, 3.14 and 3.15 into Equation 3.3 gives

$$CV_0 U(t) = -LC\beta\omega_0^{\,2}\cos\omega_0 t + \alpha + \beta\cos\omega_0 t \qquad \{3.16\}$$

Comparing 'cos' terms and time independent terms yields two relationships

$$CV_0 U(t) = \alpha \quad \text{and} \quad 0 = \left(\beta - LC\beta\omega_0^{\,2}\right) \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} \qquad \{3.17\}$$

At time t = 0 there is no charge on the capacitor i.e q = 0. Substituting for α into Equation 3.14 gives

$$\left.\begin{aligned} &0 = CV_0 U(t) + \beta.1 \qquad t = 0 \\ &\therefore \quad \beta = -CV_0 U(t) \end{aligned}\right\} \qquad \{3.18\}$$

The response of the system to a step change in input voltage is therefore

$$q(t) = CV_0\left(1 - \cos\frac{t}{\sqrt{LC}}\right)U(t) \qquad \{3.19\}$$

This waveform (see Figure 3.2) does not decay away to a final steady state value but oscillates between 0v and $2CV_0$v. In a practical LC circuit there is some resistance (primarily in the leads and inductance) and this will act to steadily reduce the peak-peak amplitude. This is known as *damping* and in the practical case of an LC circuit where the amplitude decays away slowly it is known as a *lightly damped* circuit.

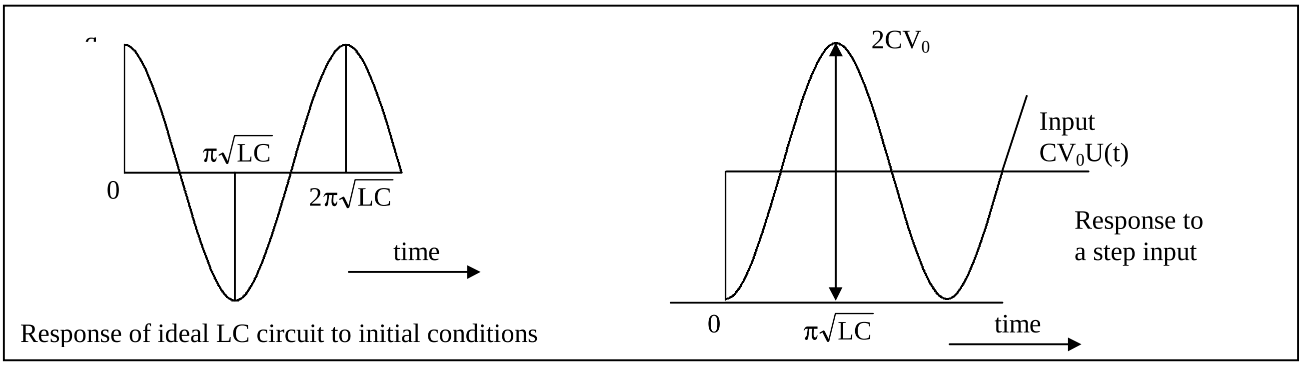


Figure 3.2 Responses of an ideal LC circuit

4.0 The Differential Operator 'p'

The use of an operator is one method of simplifying the mathematical operations required in order to solve the differential equations that can be derived from circuit based models. Effectively the introduction of an operator moves the equation out of the time domain and into a new 'space'. Another approach is to use a 'transform'; the Laplace transform moves the problem into the 's' plane whereas the Fourier transform moves the problem into the frequency domain. The use of transforms will be considered next year and for the moment only the use of an operator will be addressed. The advantage of moving into p space is that the need for integration/differentiation is removed, instead polynomial expressions are created that are far simpler to manipulate and solve.

By definition

$$p \equiv \frac{d}{dt} \qquad \{4.1\}$$

Therefore

$$p^n \equiv \frac{d^n}{dt^n} \quad \text{and} \quad p^{n-1} \equiv \frac{d^{n-1}}{dt^{n-1}} \qquad \{4.2\}$$

The action of dividing by p, i.e moving from p^n to p^{n-1}, is equivalent to integration and consequently

$$\frac{1}{p} \equiv \int dt \qquad \{4.3\}$$

Consider the generalised form of our first order equation:

$$y = x + T\frac{dx}{dt} \qquad \{2.13\}$$

Using p notation

$$y = x + Tpx \quad \therefore \; y = x(1+Tp) \qquad \{4.4\}$$

Where y is the input, x is the output and T is the system time constant. For systems of this type we know that the system response, x(t), is of the form $Ae^{-t/T}$. Using p notation we can define a *Transfer Function* for the system. The transfer function is defined as the output expressed in p divided by the input expressed in p. For a first order system

$$\frac{x}{y} = \frac{1}{1+Tp} \qquad \{4.5\}$$

Multiplying the transfer function by any input expressed in terms of p will generate the requisite output also expressed in terms of p. However, because most of the time, only the response to a few standard inputs is required (step, ramp, impulse or cosine function) it is easier to remember the time based form of any response. Inspection of Equation 4.5 clearly reveals that any response is dependent on the time constant, T, which is a term of the denominator of the transfer function. The *characteristic equation* of a transfer function is its denominator put equal to zero. The roots of the characteristic equation define the system response. The locations of these roots in p space are known as *poles*. For the transfer function defined in Equation 4.5, the characteristic equation is

$$1 + Tp = 0 \qquad \{4.6\}$$

The system has a single real pole at

$$p = -\frac{1}{T} \qquad \{4.7\}$$

This indicates that the system response will be of the form

$$x(t) = Ae^{-\frac{t}{T}} \qquad \{4.8\}$$

Consider the case of simple harmonic motion, where

$$y = \frac{d^2x}{dt^2} + \omega_0{}^2 x \qquad \{4.9\}$$

Using p notation

$$y = x\left(p^2 + \omega_0{}^2\right) \qquad \{4.10\}$$

Therefore, the system transfer function is

$$\frac{x}{y} = \frac{1}{p^2 + \omega_0{}^2} \qquad \{4.11\}$$

The characteristic equation is

$$p^2 + \omega_0{}^2 = 0 \qquad \{4.12\}$$

The system has two complex poles at

$$p = \pm j\omega_0 \qquad \{4.13\}$$

Indicating that the response is of the form

$$x(t) = A\cos\left(\omega_0 t\right) \qquad \{4.14\}$$

Strictly speaking the response is exponential in form but simplifies to Equation 4.14. It is worth remembering Euler's Identity as this is very useful when solving problems involving complex exponentials:

$$\begin{aligned} \sin\left(n\omega_0 t\right) &= \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \\ \cos\left(n\omega_0 t\right) &= \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \end{aligned} \qquad \{4.15\}$$

Now consider the case for damped harmonic motion, it is reasonable to assume that the response of the system will be some combination of Equations 4.8 and 4.14.

The generalised form of a damped second order system is

$$y = \frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0 x \qquad \{4.16\}$$

where ζ (zeta) is the coefficient of damping. If ζ is zero then the system will exhibit simple harmonic motion and any oscillation will not decay away, as ζ increases towards 1 the system is said to be *underdamped* and its output will be oscillatory. If ζ is equal to 1 then the system is *critically damped* and will respond to a step input without any oscillation. For values of ζ greater than 1 the system is *overdamped* and will respond without oscillation but the rate of response decreases as ζ increases. The undamped natural frequency, ω_0, is the frequency that the system oscillates at if ζ is zero.

Taking Equation 4.16 and applying p notation gives

$$y = x\left(p^2 + 2\zeta\omega_0 p + \omega_0{}^2\right) \qquad \{4.17\}$$

Therefore the transfer function of a second order system is

$$\frac{x}{y} = \frac{1}{p^2 + 2\zeta\omega_0 p + \omega_0{}^2} \qquad \{4.18\}$$

and the characteristic equation is

$$p^2 + 2\zeta\omega_0 p + \omega_0{}^2 = 0 \qquad \{4.19\}$$

This factorises to

$$\left(p + \zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right)\left(p + \zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}\right) = 0 \qquad \{4.20\}$$

Therefore the poles are at

$$p = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} \qquad \{4.21\}$$

And the response is of the form

$$Ae^{-\left(\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right)t} + Be^{-\left(\zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}\right)t} \qquad \{4.22\}$$

This can be rearranged (more later) into

$$Ce^{-\zeta\omega_0 t}\left(e^{j\omega_0\sqrt{1-\zeta^2}\,t} + e^{-j\omega_0\sqrt{1-\zeta^2}\,t}\right) \qquad \{4.23\}$$

The first exponential term acts as an envelope, whereas the term in brackets is a cosine function of frequency, ω_r, where

$$\omega_r = \omega_0\sqrt{1-\zeta^2} \qquad \{4.23\}$$

ω_r is referred to as the damped frequency of oscillation ($rads^{-1}$).

The three characteristic equations studied yield different poles and these can be plotted in terms of their real and imaginary components. The location of the poles in the p plane determine the system response, Figure 4.1 shows the pole locations for a damped second order system (Equation 4.21) and also the effect on pole location for constant ω_0 but varying ζ.

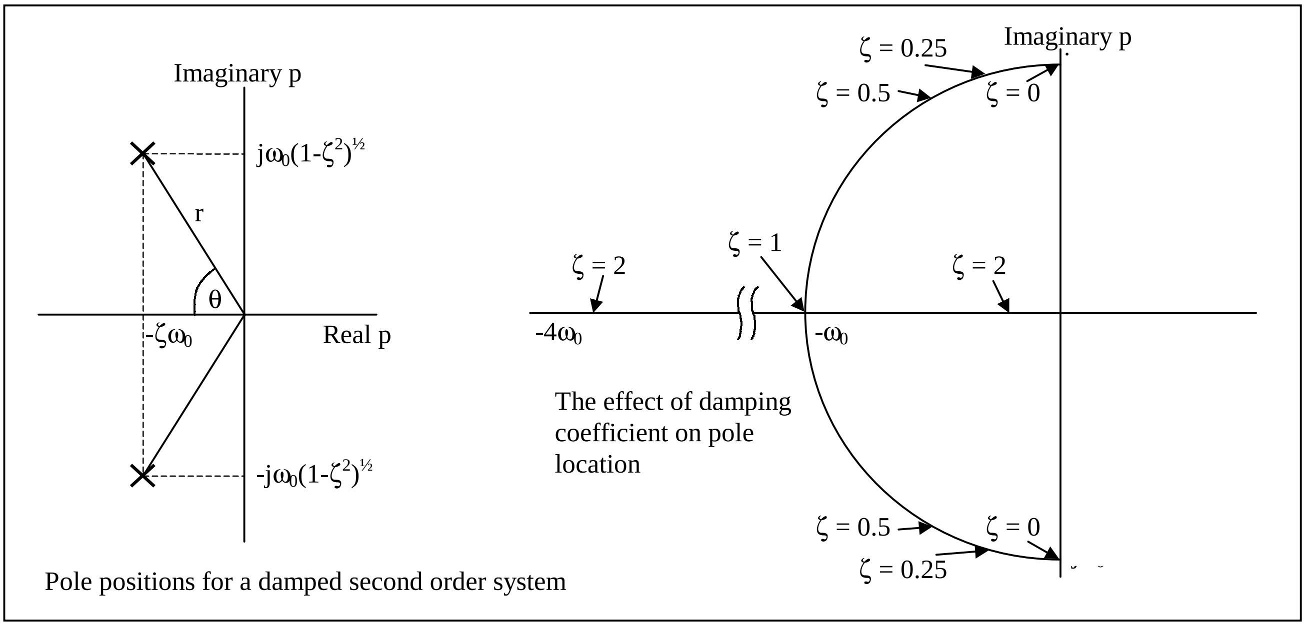


Figure 4.1 Characteristic pole locations

With reference to Figure 4.1, the distance, r, from the damped second order pole to the origin of the p plane is

$$r = \sqrt{\zeta^2\omega_0^{\ 2} + \omega_0^{\ 2}\left(1-\zeta^2\right)} = \omega_0 \qquad \{4.24\}$$

and the angle θ between the pole and the origin is

$$\theta = \cos^{-1}\left(\frac{\zeta\omega_0}{r}\right) = \cos^{-1}\zeta \qquad \{4.25\}$$

If ω_0 is held constant then a locus can be plotted in the p plane showing the effect of varying ζ on pole location. As ζ is increased from 0 the two pole positions move along the circumference of a circle radius ω_0. They meet at the negative real axis when ζ is unity. Increasing ζ beyond 1 causes the two poles to separate; one travels along the real axis towards 0 and the other travels along the real axis towards minus infinity.

4.1 Block Diagrams

The use of the p operator also allows the use of block diagrams. Block diagrams have several advantages, they allow complex structures to be represented as a network of simple blocks which can be combined to give an overall transfer function. An input to a block is multiplied by the block transfer function to generate an output. Summing junctions allow signals to be added or subtracted between blocks. The general rules for block diagrams are detailed in Table 4.1.

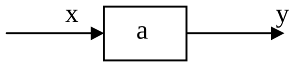
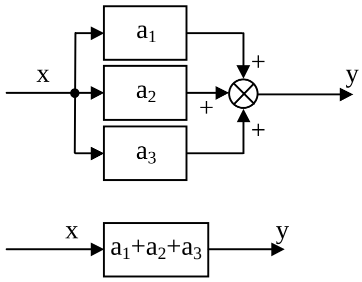
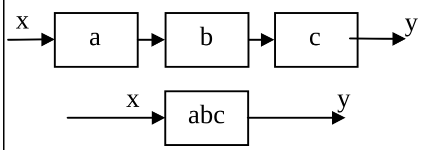
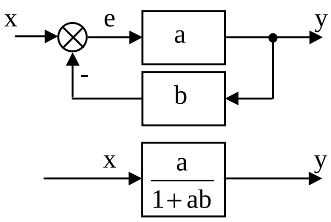
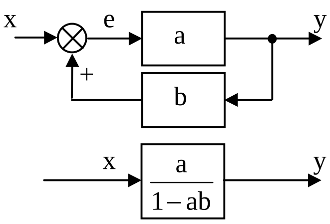
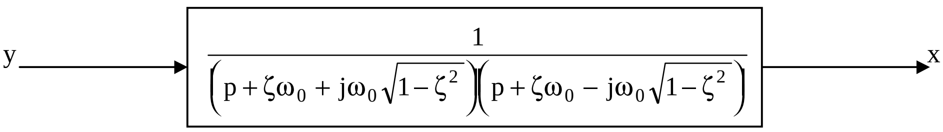
Definition	Equations	Block Diagram
Transfer Function: The transfer function is written inside a block.	y = ax	
Summed Terms: Expressions that contain a series of summed terms can be represented as separate blocks connected by a 'summing junction' or by factorising and including the summed terms within a single block.	$y = a_1x + a_2x + a_3x$ is equivalent to $y = (a_1 + a_2 + a_3)x$	
Cascaded Elements: Cascaded blocks can be multiplied together to create a single block	y = abcx = bacx = bcax etc is equivalent to	
Negative Feedback: It is common for systems to feedback signals from one block's output to its input. In many cases the output is subtracted from the desired input in order to generate an error signal. This is negative feedback.	y = ea e = x – by y = ax – aby y(1 + ab) = ax $\frac{y}{x} = \frac{a}{1+ab}$	
Positive Feedback: Much rarer is the case where the output is added to the input. The major application of positive feedback is by heavy metal guitarists, who disguise their musical shortcomings by making the audience's ears bleed.	y = ea e = x + by y = ax + aby y(1 - ab) = ax $\frac{y}{x} = \frac{a}{1-ab}$	

Table 4.1 The rules for block diagrams

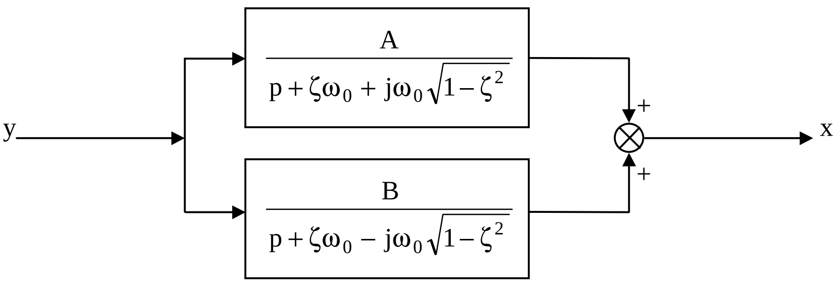
5.0 Damped Second Order Systems

Returning to the analysis of damped second order systems, it is useful to use block diagram notation in order to analyse the system response. Putting Equation 4.18 into block diagram form gives:



{5.1}

Using the rules detailed in Table 1 this block can be replaced by two blocks in parallel with a common input and whose outputs are summed together (i.e. use Partial Fractions – more later)



{5.2}

It is necessary to determine A and B. Equating Equation 5.1 with 5.2 yields

$$1 = A\left(p + \zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}\right) + B\left(p + \zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right) \qquad \{5.3\}$$

Comparing terms of p on both sides of the equation gives

$$C'p' \quad \Rightarrow \quad 0 = A + B \quad \therefore \quad B = -A \qquad \{5.4\}$$

Substituting for B in Equation 5.3 simplifies the expression to

$$1 = -2Aj\omega_0\sqrt{1-\zeta^2} \qquad \{5.5\}$$

Therefore

$$A = \frac{j}{2\omega_0\sqrt{1-\zeta^2}}, \quad B = -\frac{j}{2\omega_0\sqrt{1-\zeta^2}} \qquad \{5.6\}$$

Given that both blocks of Equation 5.2 look very similar, it is only necessary to consider one of them, e.g.

$$\longrightarrow \boxed{\frac{A}{p + \zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}}} \longrightarrow \qquad \{5.7\}$$

This is a first order block and can be compared to

$$\longrightarrow \boxed{\frac{1}{1+Tp}} \longrightarrow \qquad \{5.8\}$$

It has been established that the response of a first order block is of the form $e^{-t/T}$. Therefore Equation 5.7 needs to be rearranged to fit the form of Equation 5.8; this will yield the Time constant and allow the form of the response to be determined. Rearranging Equation 5.7 gives

$$\frac{\dfrac{A}{\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}}}{1 + \dfrac{p}{\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}}} \qquad \{5.9\}$$

Therefore the Time constant for one block is

$$T = \frac{1}{\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}} \qquad \{5.10\}$$

and the response of one block is

$$e^{-\left(\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right)t} \qquad \{5.11\}$$

This was stated earlier (Equation 4.22) based on knowledge of the roots of the characteristic equation.

5.1 Response of a damped second order system to a step input

It is known (and it is a very useful fact to recall under exam conditions) that the step response of a first order block can be described as

$$U(t) \longrightarrow \boxed{\frac{k}{1+Tp}} \longrightarrow k(1 - e^{-t/T})U(t) \qquad \{5.12\}$$

Using the same approach as above (i.e. make both blocks of Equation 5.2 match the block of Equation 5.12), the total response is

$$\left\{\frac{A}{\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}}\left(1 - e^{-\left(\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right)t}\right) + \frac{B}{\zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}}\left(1 - e^{-\left(\zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}\right)t}\right)\right\}U(t) \qquad \{5.13\}$$

The two 'k' terms (see Eqn 5.12) need to be evaluated. Consider A first, substituting Equation 5.6 for A gives

$$\frac{A}{\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}} = \frac{j}{2\omega_0\sqrt{1-\zeta^2}} \cdot \frac{1}{\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}} \qquad \{5.14\}$$

Multiplying (top and bottom) by the complex conjugate of the denominator simplifies the expression to

$$\frac{\sqrt{1-\zeta^2}+j\zeta}{2\omega_0{}^2\sqrt{1-\zeta^2}} \qquad \{5.15\}$$

Repeating the process for the constant term containing B yields the complex conjugate of Equation 5.15, therefore the total response is

$$\left\{\frac{\sqrt{1-\zeta^2}+j\zeta}{2\omega_0{}^2\sqrt{1-\zeta^2}}\left(1-e^{-\left(\zeta\omega_0+j\omega_0\sqrt{1-\zeta^2}\right)t}\right)+\frac{\sqrt{1-\zeta^2}-j\zeta}{2\omega_0{}^2\sqrt{1-\zeta^2}}\left(1-e^{-\left(\zeta\omega_0-j\omega_0\sqrt{1-\zeta^2}\right)t}\right)\right\}U(t) \qquad \{5.16\}$$

Expanding this expression gives a response of

$$\frac{1}{\omega_0{}^2}-\frac{e^{-\zeta\omega_0 t}}{\omega_0{}^2}\left\{\frac{1}{2}\left(e^{-j\omega_0\sqrt{1-\zeta^2}\,t}+e^{j\omega_0\sqrt{1-\zeta^2}\,t}\right)+\frac{j\zeta}{2\sqrt{1-\zeta^2}}\left(e^{-j\omega_0\sqrt{1-\zeta^2}\,t}-e^{j\omega_0\sqrt{1-\zeta^2}\,t}\right)\right\} \qquad t\geq 0 \qquad \{5.17\}$$

This can be further simplified using Euler's Identities to give

$$\frac{1}{\omega_0{}^2}-\frac{e^{-\zeta\omega_0 t}}{\omega_0{}^2}\left(\cos\omega_0\sqrt{1-\zeta^2}\,t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_0\sqrt{1-\zeta^2}\,t\right) \qquad t\geq 0 \qquad \{5.18\}$$

The response to a step input consists of four components.

- An offset of ω_0^{-2}, which is the steady state response as t tends to infinity
- An exponential envelope function, which dampens the oscillations
- An oscillation having a frequency of $\omega_0(1-\zeta^2)^{0.5}$
- A phase shift (the sine term)

It is possible to determine both ω_0 and ζ from a transfer function by comparison with the characteristic equation of the generalised form of a second order system i.e

$$p^2+2\zeta\omega_0 p+\omega_0{}^2=0 \qquad \{4.19\}$$

Having determined these values sketching the step response is straightforward (Figure 5.1)

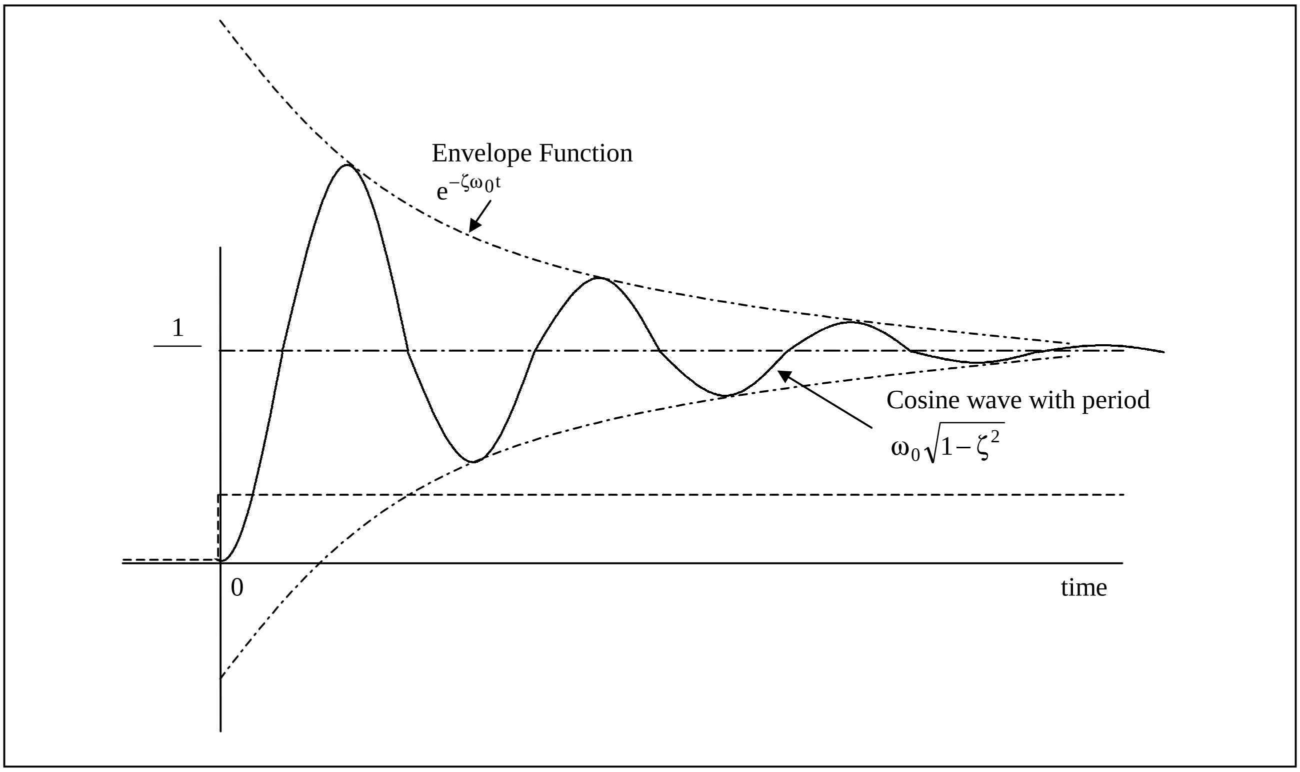


Figure 5.1 Step response of a damped second order system

6.0 Evaluating the response of systems of any order

Using the approach adopted in Section 5 it is possible to evaluate the response of higher order systems. This method can be best detailed in the form of a flow chart (Figure 6.1).

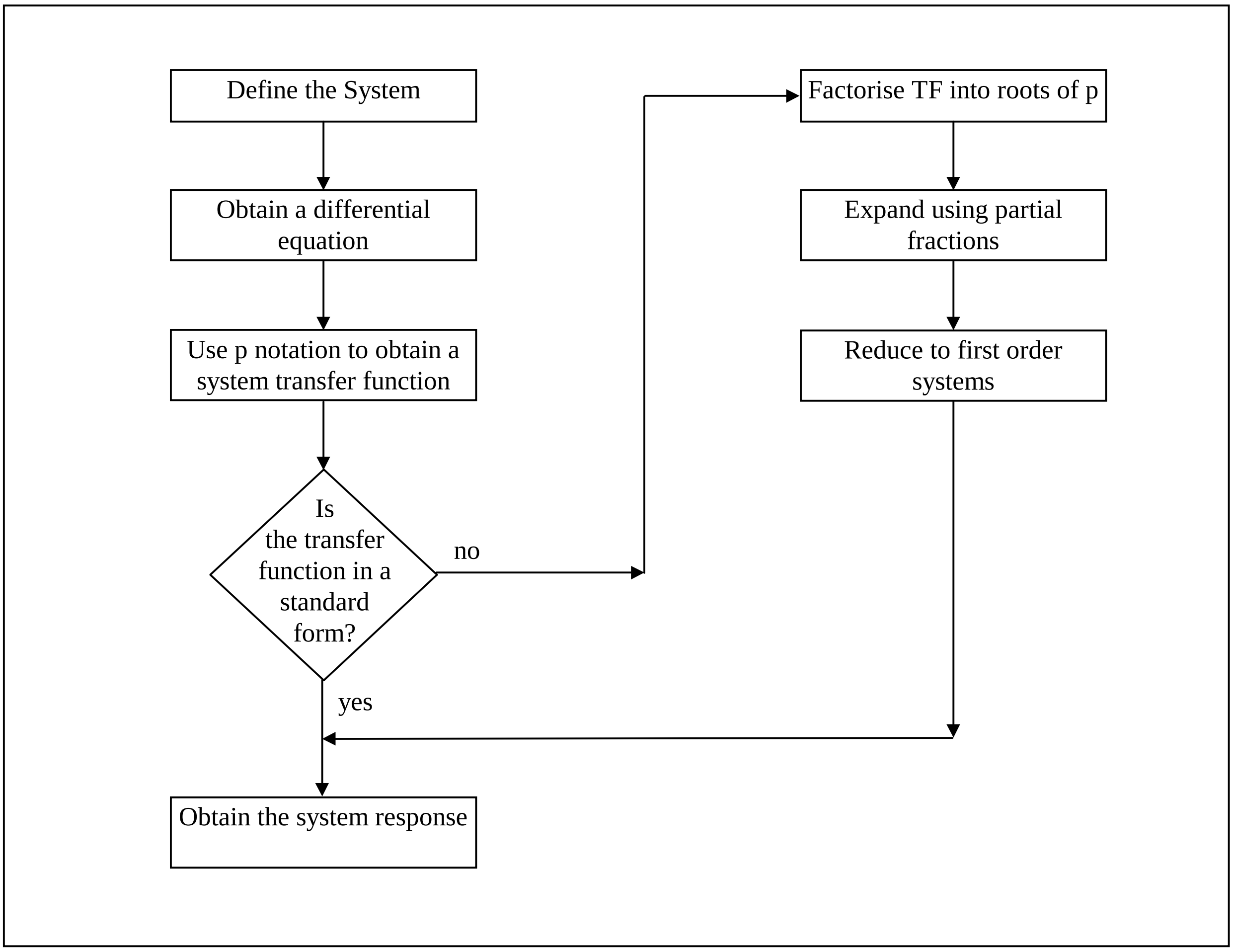


Figure 6.1 How to determine the response of a system of any order

The ability to expand a transfer function using partial fractions is important. The response for each first order block can then be evaluated and the total response will be the sum of the block responses. For the sake of completeness the rules for partial fractions are included below.

6.1 Rules for partial fractions

If the numerator is greater than or equal to the order of the denominator then use long division to obtain a polynomial in p and a proper fraction. For example, if the transfer function is

$$\frac{3p^3 + 26p^2 + 63p + 38}{p^2 + 8p + 15} \qquad \{6.1\}$$

Then evaluate the integer and remainder using long division, i.e.

$$\begin{array}{r}
3p + 2 \\
p^2 + 8p + 15 \overline{)\, 3p^3 + 26p^2 + 63p + 38} \\
-\left(\underline{3p^3 + 24p^2 + 45p}\right) \\
2p^2 + 18p + 38 \\
-\left(\underline{2p^2 + 16p + 30}\right) \\
2p + 8
\end{array} \qquad \{6.2\}$$

Therefore

$$\frac{3p^3 + 26p^2 + 63p + 38}{p^2 + 8p + 15} = 3p + 2 + \frac{2p + 8}{p^2 + 8p + 15} \qquad \{6.3\}$$

Find the roots of the denominator of the proper fraction

$$\frac{2p + 8}{p^2 + 8p + 15} = \frac{2p + 8}{(p + 3)(p + 5)} \qquad \{6.4\}$$

Split into partial fractions

$$\frac{2p+8}{(p+3)(p+5)} = \frac{A}{(p+3)} + \frac{B}{(p+5)} \qquad \{6.5\}$$

Use the cover-up rule to determine A and B. To calculate a particular coefficient, (A or B in this case) write the proper fraction without the denominator of the partial fraction that is being evaluated. Calculate the value of the modified proper fraction when p is chosen such that the missing term in the denominator would become zero (p = -3 in the case of A and –5 for B). For example,

$$\begin{aligned} A &= \left.\frac{2p+8}{p+5}\right|_{p=-3} = \frac{2}{2} = 1 \\ B &= \left.\frac{2p+8}{p+3}\right|_{p=-5} = \frac{-2}{-2} = 1 \end{aligned} \qquad \{6.6\}$$

Therefore

$$\frac{3p^3+26p^2+63p+38}{p^2+8p+15} = 3p+2+\frac{1}{p+3}+\frac{1}{p+5} \qquad \{6.7\}$$

The cover-up rule works for both real and complex roots providing no term in the denominator is raised to a power other than one. Unfortunately, there are many examples of transfer functions that contain $(p+a)^m$ where m is 2 or 3 and a is frequently zero. Proper fractions containing such terms must be treated differently. If the proper fraction, f(p) is

$$f(p) = \frac{p^n\cdots}{(p+a)^m\left(p^w\cdots\right)} \qquad \{6.8\}$$

Expanding f(p) gives

$$\frac{A_1}{(p+a)} + \frac{A_2}{(p+a)^2} + \cdots \frac{A_m}{(p+a)^m} + \frac{B}{(p+b)} + \frac{C}{(p+c)} + \cdots \qquad \{6.9\}$$

B and C etc can be determined using the cover-up rule. Let Q_p be defined as

$$Q_p = (p+a)^m f(p) \qquad \{6.10\}$$

This can be used to evaluate the A coefficients, because

$$\begin{aligned} A_m &= \left.Q_p\right|_{p=-a} \\ A_k &= \frac{1}{(m-k)!}\left.\frac{d^{m-k}}{dp^{m-k}}Q_p\right|_{p=-a} \qquad k = 1, 2, \cdots m-1 \end{aligned} \qquad \{6.11\}$$

Not as bad as it looks, consider the following example

$$f(p) = \frac{7p^3+2p^2+32}{p^5+6p^4+8p^3} \quad \text{this gives} \quad f(p) = \frac{7p^3+2p^2+32}{p^3(p+2)(p+4)} \quad \text{giving} \quad f(p) = \frac{A_1}{p} + \frac{A_2}{p^2} + \frac{A_3}{p^3} + \frac{B}{p+2} + \frac{C}{p+4}$$

Using cover – up rule for B and C

$$B = \left.\frac{7p^3+2p^2+32}{p^3(p+4)}\right|_{p=-2} = \frac{-56+8+32}{-16} = 1, \quad C = \left.\frac{7p^3+2p^2+32}{p^3(p+2)}\right|_{p=-4} = \frac{-448+32+32}{-64(-2)} = \frac{-384}{128} = -3$$

Use Eqn 6.11 for A

$$Q_p = \frac{7p^3+2p^2+32}{p^2+6p+8} \quad \therefore \quad A_3 = \left.\frac{7p^3+2p^2+32}{p^2+6p+8}\right|_{p=0} = \frac{32}{8} = 4,$$

$$A_2 = \frac{1}{1!}\left.\frac{\left(p^2+6p+8\right)\left(21p^2+4p\right)-\left(7p^3+2p^2+32\right)(2p+6)}{\left(p^2+6p+8\right)^2}\right|_{p=0} = \left.\frac{7p^4+84p^3+180p^2-32p-192}{p^4+12p^3+52p^2+96p+64}\right|_{p=0} = \frac{-192}{64} = -3$$

$$A_1 = \frac{1}{2!}\times$$

$$\left.\frac{\left(p^4+12p^3+52p^2+96p+64\right)\left(28p^3+252p^2+360p-32\right)-\left(7p^4+84p^3+180p^2-32p-192\right)\left(4p^3+36p^2+104p+96\right)}{\left(p^4+12p^3+52p^2+96p+64\right)^2}\right|_{p=0}$$

$$A_1 = \frac{(64)(-32)-(-192)(96)}{2\left(64^2\right)} = \frac{-32+288}{128} = 2 \quad \therefore \quad f(p) = \frac{2}{p} - \frac{3}{p^2} + \frac{4}{p^3} + \frac{1}{p+2} - \frac{3}{p+4}$$

Tutorial Sheet 2: 2^{nd} Order Systems, p notation, transfer functions, step response and block diagram notation

Helpful hints and suggestions

Q1 Use KVL to obtain a differential equation relating the input voltage to the current and also write an expression for the output voltage V_c. Convert both expressions using p notation and write a transfer function. Determine the characteristic equation of the transfer function. In order to determine the damping coefficient and undamped natural frequency of the circuit compare the characteristic equation with the general form for a second order system. If the choice of output is altered does it effect the characteristic equation of the circuit? To sketch the step response refer to Figure 5.1.

Q2 Use KVL to obtain an expression for each mesh and also write an expression for the output voltage. Where necessary substitute for one mesh current in terms of another so that when eventually writing a transfer function the current is removed from both the denominator and numerator.
Note the order of p will determine the number of time varying components associated with it. Eg p^2 terms will contain 2 time varying components say L_1L_2, C_1L_1, C_1L_2, C_1C_2.

Q3 Separate the transfer functions using partial fractions rearrange each block into the first order form of Equation 5.12 sum the block responses to obtain an overall response.

Q4 The switch has been open for a long time, so the capacitor voltage will have reached a steady state value. Determine the conditions before the switch is closed. Determine the transfer function for the RLC mesh. Determine the characteristic equation and calculate the poles, P_1 and P_2. The response of the circuit will be of the form

$$i(t) = Ae^{P_1 t} + Be^{P_2 t}$$

Use the initial conditions to determine A and B

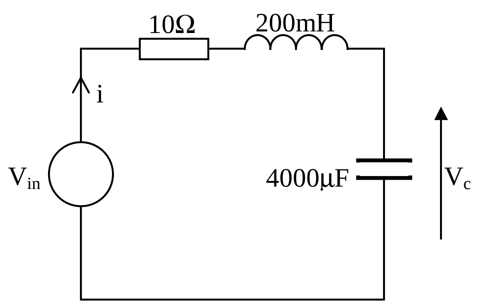
Q5 Use the rules for block diagram to reduce the diagram to a single function. Then as Q3.

1. The general form of a second order system is

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = y$$

Using p notation write a transfer function relating the input y to the output x
From the Transfer Function determine the characteristic equation
Why is it called the characteristic equation?

In the circuit below $y \equiv V_{in}$ and $x \equiv V_c$, determine ζ and ω_0 for the circuit



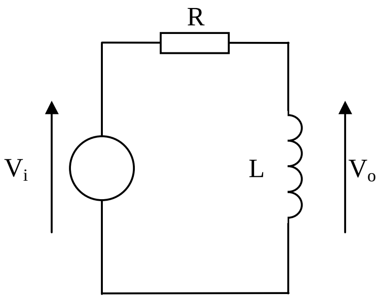
If y=0 for t<0 and y=1 for t≥0, sketch the capacitor voltage against time for t>0
If $x \equiv i$ rather than V_c will ζ and ω_0 be effected?

2. Given that a system transfer function is defined as

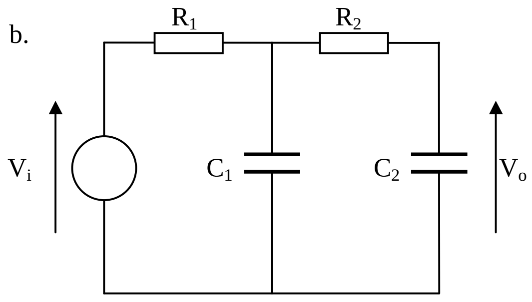
$$\frac{\text{output}(p)}{\text{input}(p)}$$

Determine transfer functions (V_o/V_i) for the following circuits

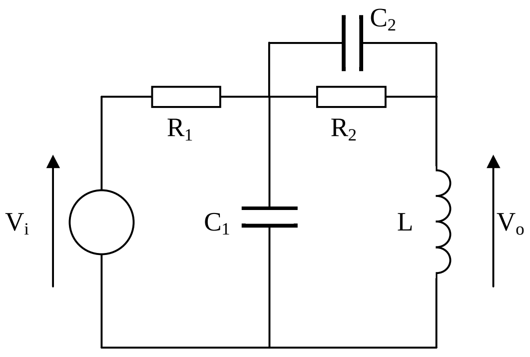
a.



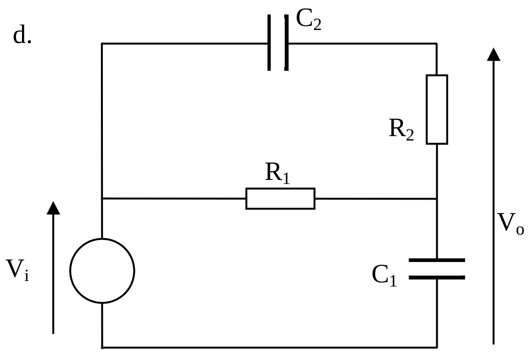
b.



c.



d.



3. Assuming zero initial conditions find the response to a step input for the following transfer functions

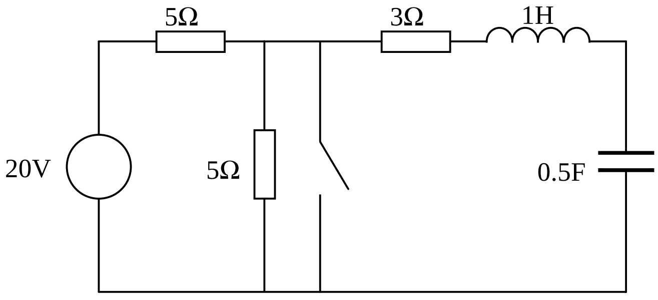
$$\frac{2p+3}{p^2+3p+2}$$

$$\frac{1}{p^2+11p+30}$$

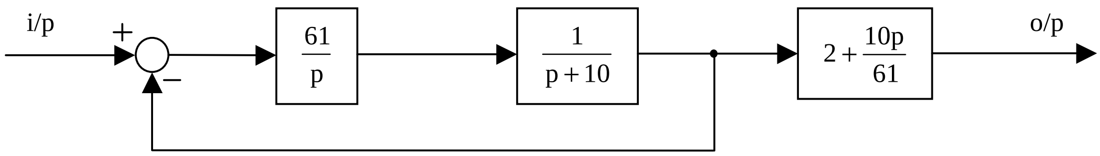
$$\frac{18p+234}{p^2+18p+117}$$

For each case sketch the system response for t>0 and determine the final steady state value

4. In the circuit below, the switch has been open for a long time. If the switch is closed at t=0 determine the current flowing through the 3Ω resistor for t>0



5. Determine the transfer function for the system shown below. From the characteristic equation or otherwise calculate the natural frequency and damping coefficient of the system.



Determine the system response to a unit step input. Sketch the output waveform and calculate the final steady state value