MAT290H1F – 2011 Application Exercise #2: *Laplace Transforms and Impulse Response*

Introduction

In the first application exercise we found that we could characterize a *RLC* circuit with a unique Green's function. This function, $K(t,u)$, was then used to find the output of that circuit for two types of input voltage functions, i.e., $v_{S1}(t) = 5R_SLCt$ V and $v_{S2}(t) = 2R_SLCe^{-\beta t}$ V.

The main advantage of the Green's function is that it does not depend on the input function, so it can be used to find the output for any type of input voltage if the circuit is linear.

This application exercise is based on the use of Laplace transforms to analyze the same linear *RLC* circuit, or network. To do this we will introduce the concept of the *impulse response* of a network, which is closely related to the Green's function for a network.

The Impulse Response

In order to properly define the impulse response of a network, we must first define the *unit impulse function,* δ(*t*). This is also called the Dirac delta function and is discussed in greater detail in Section 4.5 of your textbook. This function is defined by the two statements:

$$\delta(t-t_0) = \begin{cases} 0 & t < t_0 \\ undefined & t = t_0 \\ 0 & t > t_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

The second statement is called the *sifting* or *sampling property* of the unit impulse function because it "pulls out" the value of $f(t)$ at t_0. The peculiarity of $\delta(t-t_0)$ is that it is undefined at t_0 but when it is integrated from $-\infty$ to $+\infty$ the area under $\delta(t-t_0)$ is 1. When this was first introduced in 1927 by the physicist Paul Dirac, mathematicians were initially very sceptical about its validity as a properly defined function. Nevertheless, it was very useful in solving physical problems, which made it popular with engineers and physicists, so it gained widespread use. Eventually, in the 1940s it was properly defined by the mathematician Laurent Schwartz.

An important property of the unit impulse function is that its Laplace transform, with $t_0 = 0$, is

$$L\{\delta(t)\} = \int_0^{\infty} e^{-st}\delta(t)dt = e^{-s(0)} = 1$$

which follows from the sampling property. With this property, we can now define the impulse response of a linear network:

> *The impulse response of a linear network, $h(t)$, is the "output" of the network when the "input" of the network is the unit impulse function, $\delta(t)$, and all initial conditions are zero.*

Given this, the impulse response for the *RLC* network of Application Exercise #1 is related to the Green's function for that network through this equation, since $f(u) = \delta(u)$ in this case:

$$i_{Lp}(t) = h(t) = \int_0^t K(t,u)\left[\frac{\delta(u)}{R_SLC}\right]du$$

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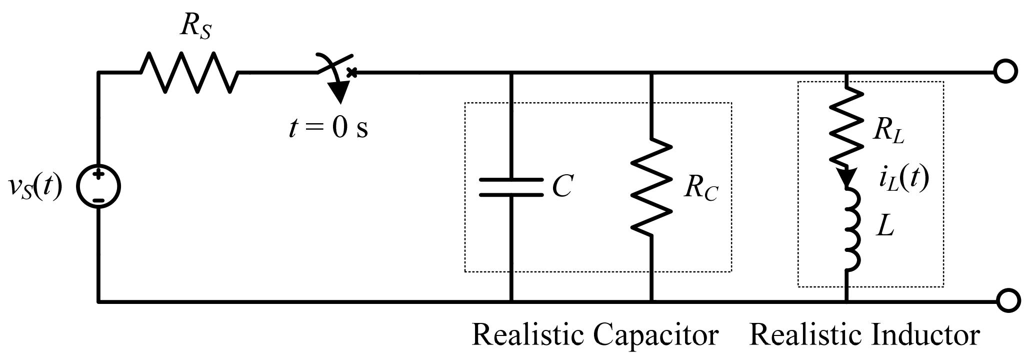
Application Exercise #2: *Laplace Transforms*

As a concrete example, let us reconsider a modified version of the *RLC* circuit of Application Exercise #1:



we will define the "output" of this *RLC* network as the current through the realistic inductor, $i_L(t)$, and the "input" is the voltage source, $v_S(t)$.

We can now characterize this *RLC* network with its impulse response, $h(t)$, by finding the output $i_L(t) = h(t)$ when the input is the unit impulse function, $v_S(t) = \delta(t)$. In essence, the impulse response tells us how this circuit behaves if we turn the voltage source on and off in an infinitesimally small amount of time. Or in other words, it gives the current $i_L(t)$ if the circuit is subjected to an extremely short burst of energy.

(a) One method which can be used to find the impulse response of a network is to "excite" the network with the unit impulse function and then solve the resulting initial-value problem. Therefore, for this *RLC* circuit, use Laplace transform techniques to solve the initial-value problem given by:

$$i_L'' + Ai_L' + Bi_L = D\delta(t) \quad i_L(0) = 0\text{ A}, i_L'(0) = 0\text{ A/s} \qquad \text{Eqn. 1}$$

where $A = \left(\frac{R_L}{L} + \frac{R_S+R_C}{R_S R_C C}\right)$, $B = \left[\frac{1}{LC} + \frac{(R_S+R_C)R_L}{R_S R_C LC}\right]$, and $D = \frac{1}{R_S LC}$, to find $I_L(s) = H(s) = \mathcal{L}\{h(t)\}$. Note that the input source is the unit impulse function, and that the initial conditions are zero, which are the two key requirements needed to find the impulse response $h(t)$.

Find both the impulse response of this circuit, $h(t)$, and its Laplace transform, $H(s)$. Your answer for $H(s)$ should be expressed in terms of A, B, and D. You can express your answer for $h(t)$ in terms of D and the roots of the denominator of $H(s)$, p_1 and p_2. Make sure you clearly state how these roots p_1 and p_2 relate to A and B.

Application Exercise #2: *Laplace Transforms (cont'd)*

Part (a) (continued)

Application Exercise #2: *Laplace Transforms (cont'd)*

(b) The Laplace transform of the impulse response, $H(s)$, is called the *transfer function* of a network. This is because in the s-domain, it relates *any* input voltage function, $V_S(s)$, to the output current function, $I_L(s)$, i.e., $I_L(s) = H(s)\,V_S(s)$, *as long as the initial conditions are zero*. This means that in the time-domain the output is given as the convolution of the impulse response and the input signal, i.e., $i_L(t) = h(t) * v_s(t) = \int_0^t h(u)v_s(t-u)du$.

Now consider the most general case, where *the initial conditions are no longer zero*, which means that the initial-value problem is now:

$$i_L'' + Ai_L' + Bi_L = Dv_S(t) \quad i_L(0) = E \text{ A}, i_L'(0) = F \text{ A/s} \qquad \text{Eqn. 3}$$

where E and F are real constants and A, B, and D are the same as defined above

Using Eqn. 3, show that the output, $I_L(s)$, can be written as:

$$I_L(s) = G(s)H(s) + Q(s)H(s)$$

where $G(s)H(s)$ represents the *zero-state* response, which relates only to case of zero initial conditions but with the independent input source, $V_S(s)$, connected. While $Q(s)H(s)$ represents the *zero-input* response, which relates only to the initial "state" of the circuit (i.e., the initial conditions) but with the independent input source not connected. Clearly identify $G(s)$ and $Q(s)$, and the zero-state and zero-input parts of the s-domain response.

Application Exercise #2: *Laplace Transforms (cont'd)*

Part (b) (continued)

Application Exercise #2: *Laplace Transforms (cont'd)*

(c) In practical inductors, the resistance of the inductor, R_L, depends on the value of the inductance, L. Generally, this is an direct proportion relationship and for this particular circuit the inductor's resistance is given by $R_L = 1 \times 10^8 L^2\ \Omega$.

Use MATLAB to determine the required value of L, such that the complementary solution (i.e., the solution to the homogeneous differential equation: $i_L'' + Ai_L' + Bi_L = 0$) is given as:

$$i_{Lc}(t) = e^{-1005t}[c_1 \cos(7.071 \times 10^4 t) + c_2 \sin(7.071 \times 10^4 t)]$$

To do this, determine how the poles of the transfer function $H(s)$ for this *RLC* network depend on the inductance L, and how these poles relate to the complementary solution given above. To support your conclusion include:

i) A plot from MATLAB of the real parts of the two poles of $H(s)$ versus the inductance L and,
ii) A plot from MATLAB of the imaginary parts of the two poles of $H(s)$ versus the inductance L.

Your plots should cover the range of $1\ \mu\text{H} \le L \le 1\ \text{mH}$, and the other circuit elements are given by $R_S = 50\ \Omega$, $R_C = 5000\ \Omega$, and $C = 20\ \mu\text{F}$.

(d) Using your result for L found in part (c), determine the output current, $i_L(t)$, for the following cases. You can use MATLAB to determine the poles and residues needed to find the Laplace inverse transform (as described in MATLAB module #2), but you must clearly show your partial fraction expansion for $I_L(s)$.

i) The input source is given as:

$$v_S(t) = \begin{cases} 0 & t < 0 \\ 5000t\ \text{V} & 0 \le t < 2\ \text{ms} \\ 0 & t \ge 2\ \text{ms} \end{cases}$$

and the circuit is initially uncharged, meaning that $i_L(0+) = 0$ A and $i_L'(0+) = 0$ A/s.

ii) The input source is the exact same as given above in part i), but the circuit has the initial conditions given by $i_L(0+) = 0.5$ A and $i_L'(0+) = 0$ A/s (this initial condition comes from a source which is not shown in the circuit and is disconnected at $t = 0$). For this case, use the *plotyy* or the *subplot* command in MATLAB to create a plot of the input $v_S(t)$ and the output $i_L(t)$ on the same figure over the time range of $0\ \text{s} \le t \le 5\ \text{ms}$. Make sure this plot is your own. To solve this part, it will help to make use of your results from part (b).

<u>Application Exercise #2</u>: *Laplace Transforms (cont'd)*

Parts (c) and (d) (continued)

<u>Application Exercise #2</u>: *Laplace Transforms (cont'd)*

Parts (c) and (d) (continued)

By signing below, I declare that this work is entirely my own.

Signature: ____________________ Date: _________________