MAT290H1F – 2011 Application Exercise #1: *Green's Functions*

The variation of parameters technique allows the very powerful concept of Green's functions to be easily developed. Consider the initial-value problem based on the general 2nd-order linear nonhomogeneous differential equation:

$$y'' + P(t)y' + Q(t)y = f(t) \qquad t \in I,\ y(t_0) = A, y'(t_0) = B \qquad \text{Eqn. (1)}$$

where $P(t)$, $Q(t)$, and $f(t)$ are continuous on the interval I. According to the variation of parameters, a particular solution to this equation can be written as

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(u)f(u)}{W[y_1, y_2](u)}du + y_2(t)\int_{t_0}^{t} \frac{y_1(u)f(u)}{W[y_1, y_2](u)}du$$

with $y_1(t)$ and $y_2(t)$ coming from the complementary solution, $y_c(t)$, and u being a dummy variable of integration. Note, we have taken equation (5) on pg. 129 of your textbook and solved for $u_1(t)$ and $u_2(t)$ directly through integration. Also, we have replaced the indefinite integrals with definite ones which ensure that the initial conditions for $y_p(t)$ are $y_p(t_0) = 0$ and $y_p'(t_0) = 0$. The original initial conditions of the problem,$y(t_0) = A, y'(t_0) = B$, are then satisfied *entirely* by the complementary part of the general solution. The term $W[y_1, y_2](u)$, is the Wronskian of the two functions $y_1(u)$ and $y_2(u)$.

We can rewrite this expression to be:

$$y_p(t) = \int_{t_0}^{t} \frac{y_1(u)y_2(t) - y_2(u)y_1(t)}{W[y_1, y_2](u)} f(u)du$$

or

$$y_p(t) = \int_{t_0}^{t} K(t, u)f(u)du$$

where $K(t, u) = \frac{y_1(u)y_2(t) - y_2(u)y_1(t)}{W[y_1, y_2](u)}$. The function $K(t, u)$ is called the *Green's function* for this differential equation. The key thing to observe is that the Green's function is independent of the forcing function$f(t)$, as it *only* depends on the associated homogeneous differential equation!

The use of Green's functions results in two key ideas:

(a) Consider the initial-value problem given by Eqn. (1). If the complementary solution, $y_c(t) = c_1y_1(t) + c_2y_2(t)$, and the Green's function, $K(t, u)$, are known, then the complete solution for *different* forcing functions can simply be found through integration, i.e.:

$$y(t) = y_c(t) + y_p(t) = c_1y_1(t) + c_2y_2(t) + \int_{t_0}^{t} K(t, u)f(u)du$$

(b) If one *changes the initial conditions* for the initial value problem given by Eqn. (1), but *keeps the forcing function the same*, then to find the new solution all that is required is to find the new coefficients c_1 and c_2 since the initial conditions of the particular solution are always zero, i.e., $y_p(t_0) = 0$ and $y_p'(t_0) = 0$. The particular solution remains unchanged since this only depends on the Green's function and the forcing function.

For an electric circuit, you can think of the Green's function as being directly related to the forced response of that circuit for a source of unit amplitude (i.e., the forcing function, $f(t) = 1$, thus $y_p(t) = \int_{t_0}^{t} K(t, u)du$). In *ECE216H1S: Signals and Systems* you will learn about the concept of the *impulse response* of a system, which is closely related to the Green's function. This is a very powerful concept for circuit and system design.

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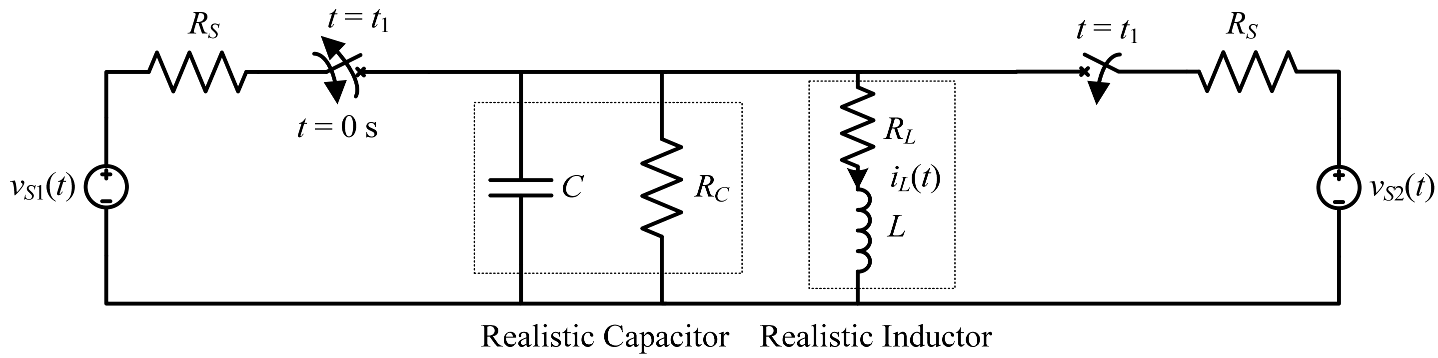
The circuit given below represents a band-pass filter which is often used in wireless communication systems. The inductor and capacitor in this circuit include the internal resistances which characterize the losses associated with the inductor's coil (R_L) and the capacitor's insulator (dielectric) (R_C). Thus, this is a more realistic model of the parallel *RLC* circuit.



The circuit is excited by two voltage sources, with the first one, $v_{S1}(t) = 5R_SLCt$ Volts being connected to the circuit at $t = 0$ s, and the second one, $v_{S2}(t) = 2R_SLCe^{-\beta t}$ Volts being connected at a later time $t = t_1$. Note that the first source, $v_{S1}(t)$ is disconnected from the circuit at the instant that the second source is connected, i.e., $t = t_1$.

(a) Show that for this parallel *RLC* circuit, the inductor current $i_L(t)$ is governed by the differential equation

$$i_L'' + \left(\frac{R_L}{L} + \frac{R_S + R_C}{R_SR_CC}\right) i_L' + \left[\frac{1}{LC} + \frac{(R_S + R_C)R_L}{R_SR_CLC}\right] i_L = \frac{1}{R_SLC} v_{Sx}(t)$$

where $v_{Sx}(t)$ represents the voltage source which is connected to the circuit, i.e., $v_{Sx}(t) = v_{S1}(t)$ for $0 \le t < t_1$ and $v_{Sx}(t) = v_{S2}(t)$ for $t_1 \le t < \infty$.

(b) If it is known that the complementary solution for the differential equation of part (a) is given by:

$$i_{Lc}(t) = e^{-\alpha t}(c_1 + c_2 t)$$

where α is a positive real constant, show that the Green's function, $K(t,u)$, for the differential equation of part (a) is:

$$K(t,u) = e^{-\alpha(t-u)}(t - u)$$

Application Exercise #1: *Green's Functions (con't)*

Parts (a) and (b) (cont'd)

Application Exercise #1: *Green's Functions (con't)*

(c) Use the Green's function given in part (b) to determine the particular solution for the inductor current for the time interval $0 \leq t < t_1$. You may assume that the capacitor and the inductor are initially "uncharged" at $t = 0$ s, meaning that $i_L(0+) = 0$ A and $i_L'(0+) = 0$ A/s.

Application Exercise #1: *Green's Functions (con't)*

(d) Use the Green's function given in part (b) to determine the particular solution for the inductor current for the time interval $t_1 \le t < \infty$, where $t_1 = \frac{2}{\alpha}$.

(e) Use the Method of Undetermined Coefficients to verify your expression from part (c) for the inductor current for the time interval $0 \le t < t_1$, where $t_1 = \frac{2}{\alpha}$. Note that the differential equation can be written as: $i_L'' + Ai_L' + Bi_L = Cv_{Sx}(t)$, where A, B, and C represent the coefficients in terms of the circuit element values. If you determine the relationship between A and α, and B and α you can verify the expression you have found in part (c). *Hint*: Pay close attention to the type of complementary solution that this differential equation has for this circuit, which was given in part (b).

(f) Now consider the case in which $R_L = 100\ \Omega$, $R_C = 380\ \Omega$, $R_S = 50\ \Omega$, $L = 290$ mH, $C = 18.9$ µF, $\alpha = 772.4$ 1/s, and $\beta = 2\alpha$. Use Matlab to plot the circuit response that you found in parts (c) and (d) above from $0 \le t \le 5t_1$. Include the printout with these notes.

<u>Application Exercise #1</u>: *Green's Functions (con't)*

Parts (e) and (f) (cont'd)

By signing below, I declare that this work is entirely my own.

Signature: ___________________ Date: ________________