Lecture 5

I. Operational calculus using the Fourier transformation

In Eqs. (2.1)–(2.6), we have developed the harmonic analysis for the calculation of the frequency domain transfer function if the input parameter (scalar or vector) is harmonic, i.e. it is proportional to $\exp(\pm i\omega t)$. However, the harmonic analysis becomes too intricate if the input parameters is proportional to several $\exp(\pm i\omega_n t)$ with the different ω_n. For example, in the circuit shown in Fig. 1 (**see also Appendix 2 in Lecture 2**), we have two harmonic generators with two different frequencies.

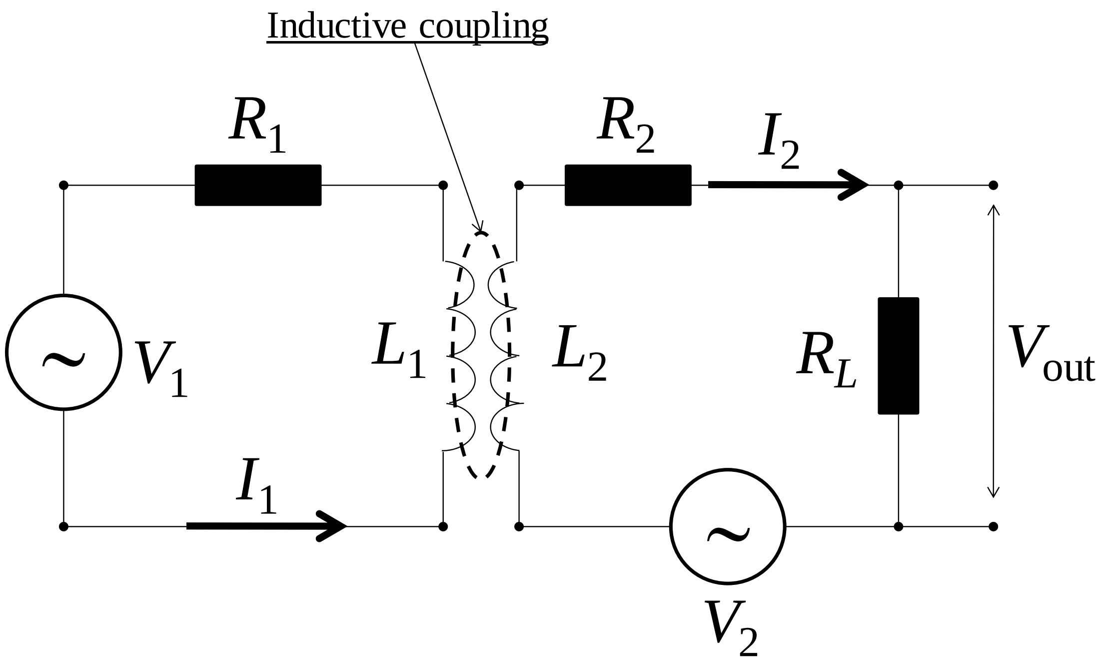


Fig. 1 Two inductively connected contours with two harmonic generators.

For this circuit, we obtain:

$$\begin{cases} R_1 I_1(t) + L_1 \dfrac{dI_1(t)}{dt} + M\dfrac{dI_2(t)}{dt} = V_1(t) \\ (R_2 + R_L) I_2(t) + L_2 \dfrac{dI_2(t)}{dt} + M\dfrac{dI_1(t)}{dt} = V_2(t) \end{cases} \tag{1}$$

where $V_1(t) = A\exp(\pm i\omega_1 t)$ and $V_2(t) = B\exp(\pm i\omega_2 t)$. To solve this system of differential equations by means of the harmonic analysis, we have to represent the currents as $I_1(t) = C_{11}\exp(\pm i\omega_1 t) + C_{12}\exp(\pm i\omega_2 t)$ and $I_2(t) = C_{21}\exp(\pm i\omega_1 t) + C_{22}\exp(\pm i\omega_2 t)$, where C_{nm} are unknown complex amplitudes (four in total). Instead of this, we could try to apply the Fourier transformation to the left and right parts of the system (1) to obtain the algebraic system for the Fourier images. This Fourier transformation must be understood as a generalised one, and we will use Eqs. (4.25)–(4.28) for this purpose:

$$\hat{f}_\varepsilon(p) = \int_{-\infty}^{0} f(x)\exp(-i\,p\,x)\exp(\varepsilon x)dx + \int_{0}^{+\infty} f(x)\exp(-i\,p\,x)\exp(-\varepsilon x)dx$$

$$f_\varepsilon(x)=\frac{1}{2\pi}\int_{-\infty}^{0}\hat{f}(p)\exp(i\,p\,x)\exp(\varepsilon\,p)dp+\frac{1}{2\pi}\int_{0}^{+\infty}\hat{f}(p)\exp(i\,p\,x)\exp(-\varepsilon\,p)dp$$

$$\hat{f}(p)=\lim_{\varepsilon\to 0}\hat{f}_\varepsilon(p)\ \text{(direct transformation)}$$

$$f(x)=\lim_{\varepsilon\to 0}f_\varepsilon(x)\ \text{(inverse transformation)}$$

The properties (3.4) and (3.5) remain true for this generalised Fourier transformation. For example, Eq. (3.4):

$$\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{0}\frac{df(x)}{dx}\exp(-ipx)\exp(\varepsilon x)dx+\int_{0}^{+\infty}\frac{df(x)}{dx}\exp(-ipx)\exp(-\varepsilon x)dx\right\}=$$

$$=\lim_{\varepsilon\to 0}\left\{\frac{df(0)}{dx}+(ip-\varepsilon)\int_{-\infty}^{0}f(x)\exp(-ipx)\exp(\varepsilon x)dx-\frac{df(0)}{dx}+(ip+\varepsilon)\int_{0}^{+\infty}f(x)\exp(-ipx)\exp(-\varepsilon x)dx\right\}=$$

$$=\lim_{\varepsilon\to 0}\left\{(ip-\varepsilon)\int_{-\infty}^{0}f(x)\exp(-ipx)\exp(\varepsilon x)dx+(ip+\varepsilon)\int_{0}^{+\infty}f(x)\exp(-ipx)\exp(-\varepsilon x)dx\right\}=$$

$$=ip\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{0}f(x)\exp(-ipx)\exp(\varepsilon x)dx+\int_{0}^{+\infty}f(x)\exp(-ipx)\exp(-\varepsilon x)dx\right\}=ip\lim_{\varepsilon\to 0}\hat{f}_\varepsilon(p)$$

And, for the generalised Fourier transformation of $\frac{d^n f(x)}{dx^n}$, we obtain: $(ip)^n\lim_{\varepsilon\to 0}\hat{f}_\varepsilon(p)$. Applying the Fourier transformation (generalised) to Eq. (1), we obtain the system of algebraic equation for the Fourier images ($p=\omega$):

$$\begin{cases}R_1\hat{I}_1(\omega)+i\omega L_1\hat{I}_1(\omega)+i\omega M\hat{I}_2(\omega)=\hat{V}_1(\omega)\\(R_2+R_L)\hat{I}_2(\omega)+i\omega L_2\hat{I}_2(\omega)+i\omega M\hat{I}_1(\omega)=\hat{V}_2(\omega)\end{cases}\tag{2}$$

This system can be written and solved in the matrix form:

$$\hat{\mathbf{R}}(i\omega)\begin{pmatrix}\hat{I}_1(\omega)\\\hat{I}_2(\omega)\end{pmatrix}=\begin{pmatrix}\hat{V}_1(\omega)\\\hat{V}_2(\omega)\end{pmatrix}\ \text{– is the generalised Ohm's law}\tag{3}$$

where

$$\hat{\mathbf{R}}(i\omega)=\begin{pmatrix}R_1+iL_1\omega & iM\omega\\ iM\omega & R_2+R_L+iL_2\omega\end{pmatrix}\ \text{– is the matrix impedance}\tag{4}$$

$$\hat{\mathbf{F}}(\omega)=\hat{\mathbf{R}}^{-1}(i\omega)=\begin{pmatrix}\hat{F}_{11}(\omega) & \hat{F}_{12}(\omega)\\ \hat{F}_{21}(\omega) & \hat{F}_{22}(\omega)\end{pmatrix}\ \text{– is the matrix transfer function in the frequency domain}\tag{5}$$

$$\begin{pmatrix}\hat{I}_1(\omega)\\\hat{I}_2(\omega)\end{pmatrix}=\hat{\mathbf{F}}(i\omega)\begin{pmatrix}\hat{V}_1(\omega)\\\hat{V}_2(\omega)\end{pmatrix}=\begin{pmatrix}\hat{F}_{11}(\omega)\times\hat{V}_1(\omega)+\hat{F}_{12}(\omega)\times\hat{V}_2(\omega)\\\hat{F}_{21}(\omega)\times\hat{V}_1(\omega)+\hat{F}_{22}(\omega)\times\hat{V}_2(\omega)\end{pmatrix}\ \text{– is the solution of (2) in the matrix form}\tag{6}$$

(We do not calculate $\hat{\mathbf{F}}(\omega)$ here)

Note that the matrix (4) can be obtained by means of the harmonic analysis with $\exp(+i\omega t)$**! Therefore, the harmonic analysis can be used to derive the frequency domain transfer function (scalar or vector).** To find the currents in the time domain, we have to apply the inverse Fourier transformation (generalised) to Eq. (6) and use Eq. (3.7):

$$\begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{11}(\omega)\times\hat{V}_1(\omega)\exp(i\omega t)d\omega + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{12}(\omega)\times\hat{V}_2(\omega)\exp(i\omega t)d\omega \\ \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{21}(\omega)\times\hat{V}_1(\omega)\exp(i\omega t)d\omega + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{22}(\omega)\times\hat{V}_2(\omega)\exp(i\omega t)d\omega \end{pmatrix} =$$

$$= \begin{pmatrix} \int\limits_{-\infty}^{t} F_{11}(t-s)V_1(s)ds + \int\limits_{-\infty}^{t} F_{12}(t-s)V_2(s)ds \\ \int\limits_{-\infty}^{t} F_{21}(t-s)V_1(s)ds + \int\limits_{-\infty}^{t} F_{22}(t-s)V_2(s)ds \end{pmatrix} \qquad (7)$$

The first "spectral" form of Eq. (7)

$$\begin{pmatrix} \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{11}(\omega)\times\hat{V}_1(\omega)\exp(i\omega t)d\omega + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{12}(\omega)\times\hat{V}_2(\omega)\exp(i\omega t)d\omega \\ \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{21}(\omega)\times\hat{V}_1(\omega)\exp(i\omega t)d\omega + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\hat{F}_{22}(\omega)\times\hat{V}_2(\omega)\exp(i\omega t)d\omega \end{pmatrix}$$

is more convenient when $V_{1,2}(t)$ **are some periodical functions. In this case, we can use Eqs. (4.42)–(45) and Eq. (4.49).** In our Eq. (1), $V_1(t) = A\exp(\pm i\omega_1 t)$ and $V_2(t) = B\exp(\pm i\omega_2 t)$, and according to Eq. (4.42), we obtain:

$$\begin{aligned} \hat{V}_1(\omega) &= 2\pi A\delta(\omega \mp \omega_1) \\ \hat{V}_2(\omega) &= 2\pi B\delta(\omega \mp \omega_2) \end{aligned} \qquad (8)$$

Putting Eq. (8) into Eq. (7), we obtain:

$$\begin{aligned} \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix} &= \begin{pmatrix} A\times\hat{F}_{11}(\pm\omega_1)\times\exp(\pm i\omega_1 t) + B\times\hat{F}_{12}(\pm\omega_2)\exp(\pm i\omega_2 t) \\ A\times\hat{F}_{21}(\pm\omega_1)\times\exp(\pm i\omega_1 t) + B\times\hat{F}_{22}(\pm\omega_2)\times\exp(\pm i\omega_2 t) \end{pmatrix} \\ V_{out}(t) &= R_L \times I_2(t) \end{aligned} \qquad (9)$$

The concrete example considered above helps us to formulate the operational calculus in its most general matrix form for any vector input $\vec{\mathbf{V}}_{in}(t)$ and output $\vec{\mathbf{V}}_{out}(t)$ parameters (compare with Eqs. (2.2*), (2.5*), and (2.6*) in Lecture 2):

$\hat{\mathbf{P}}\left(\frac{d}{dt}\right)\left[\vec{\mathbf{V}}_{out}(t)\right] = \vec{\mathbf{V}}_{in}(t)$ – is the initial system of linear differential equations describing a stationary linear network (10)

$\hat{\mathbf{F}}(\omega) = \hat{\mathbf{P}}^{-1}(+i\omega)$ – is the matrix transfer function in the frequency domain (11)

$$\vec{\mathbf{V}}_{out}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\mathbf{F}}(\omega)\hat{\vec{\mathbf{V}}}_{in}(\omega)\exp(i\omega t)d\omega$$ – is the output signal in the spectral form (12)

$$\vec{\mathbf{V}}_{out}(t)=\int_{-\infty}^{t}\hat{\mathbf{F}}(t-s)\vec{\mathbf{V}}_{in}(s)ds$$ – is the output signal in the convolution form (13)

where

- $\hat{\mathbf{P}}(t)$ is a square matrix, each element of which is a polynomial of some order
- $\hat{\mathbf{F}}(\omega)=\hat{\mathbf{P}}^{-1}(+i\omega)$ is the inverse matrix with respect to $\hat{\mathbf{P}}(+i\omega)$
- $\hat{\mathbf{P}}(+i\omega)$ is obtained from $\hat{\mathbf{P}}(t)$ by means of the harmonic analysis with $\exp(+i\omega t)$
- $\hat{\vec{\mathbf{V}}}_{in}(\omega)$ is the generalised Fourier image of $\vec{\mathbf{V}}_{in}(t)$
- $\hat{\mathbf{F}}(t>0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\mathbf{F}}(\omega)\exp(i\omega t)d\omega$ is the matrix transfer function in the time domain which must be calculated using Eqs. (4.53)–(4.55)

The following table will be useful:

Fourier transformation	**Harmonic analysis**	**Impedance analysis**
Direct: $\sim\int(\ldots)\exp(-ipt)dt$ Inverse: $\sim\int(\ldots)\exp(+ipt)dp$	$\sim\exp(+ipt)$	$Z_C=-\frac{i}{C\omega}$ $Z_L=i\omega L$
Direct: $\sim\int(\ldots)\exp(+ipt)dt$ Inverse: $\sim\int(\ldots)\exp(-ipt)dp$	$\sim\exp(-ipt)$	$Z_C=\frac{i}{C\omega}$ $Z_L=-i\omega L$

II. Transition processes

Let us demonstrate application of the operational calculus for the charge/discharge processes in a RC circuit (see (2.10)). The main steps (see above):

- $RC\frac{dV_C(t)}{dt}+V_C(t)=V(t)$ – initial differential equation, where $V_C(t)$ is the voltage across the capacitor at the time moment t
- $P(t)=RCt+1$ – characteristic polynomial in the time domain

- $\hat{P}(+i\omega) = iRC\omega + 1$ – characteristic polynomial in the frequency domain ($+i\omega$!)
- $\hat{F}(\omega) = \dfrac{1}{\hat{P}(+i\omega)} = \dfrac{1}{iRC\omega + 1} = \dfrac{1}{iRC\left(\omega - \dfrac{i}{RC}\right)}$ – transfer function in the frequency domain
- $\hat{F}(\omega)$ has only one pole $p_1 = \dfrac{i}{RC}$ of the first degree and it does not have any growing or constant part
- According to Eq. (4.55), $res[\hat{F}(p_1)] = \dfrac{1}{iRC}$
- According to Eq. (4.53), $F(t > 0) = \dfrac{1}{RC}\exp\left(-\dfrac{t}{RC}\right)$

- According to (3.8), $V_C(t) = \dfrac{1}{RC}\int\limits_{-\infty}^{t} \exp\left(-\dfrac{(t-s)}{RC}\right)V(s)ds$ (14)

Since $V_R(t) = RC\dfrac{dV_C(t)}{dt}$, we obtain:

$$V_R(t) = V(t) - \frac{1}{RC}\int\limits_{-\infty}^{t} \exp\left(-\frac{(t-s)}{RC}\right)V(s)ds \tag{15}$$

In Eq. (15), we have used the following general formula for the derivative from the integral which depends on a parameter t:

$$\frac{d}{dt}\left[\int\limits_{a(t)}^{b(t)} f(s,t)ds\right] = \int\limits_{a(t)}^{b(t)} \frac{df(s,t)}{dt}ds + \frac{db(t)}{dt}f\big(b(t),t\big) - \frac{da(t)}{dt}f\big(a(t),t\big) \tag{16}$$

If $V(t) = \begin{cases} V_1, & t \le 0 \\ V_2, & t > 0 \end{cases}$, where $V_{1,2}$ are some constants (positive or negative), using Eq. (14) we obtain:

$$\begin{cases} V_C(t \le 0) = \dfrac{V_1}{RC}\displaystyle\int\limits_{-\infty}^{t} \exp\left(-\dfrac{(t-s)}{RC}\right)ds = V_1\left[\exp\left(-\dfrac{(t-s)}{RC}\right)\right]_{s=-\infty}^{s=t} = V_1 \\ \\ V_C(t > 0) = \dfrac{V_1}{RC}\displaystyle\int\limits_{-\infty}^{0} \exp\left(-\dfrac{(t-s)}{RC}\right)ds + \dfrac{V_2}{RC}\displaystyle\int\limits_{0}^{t} \exp\left(-\dfrac{(t-s)}{RC}\right)ds = \\ = V_1\left[\exp\left(-\dfrac{(t-s)}{RC}\right)\right]_{s=-\infty}^{s=0} + V_2\left[\exp\left(-\dfrac{(t-s)}{RC}\right)\right]_{s=0}^{s=t} = \\ = V_2 + (V_1 - V_2)\exp\left(-\dfrac{t}{RC}\right) \end{cases}$$

$$\begin{cases} V_R(t \le 0) = RC\dfrac{dV_C(t)}{dt} = RC\dfrac{dV_1}{dt} \equiv 0 \\ \\ V_R(t > 0) = RC\dfrac{d}{dt}\left[V_2 + (V_1 - V_2)\exp\left(-\dfrac{t}{RC}\right)\right] = (V_2 - V_1)\exp\left(-\dfrac{t}{RC}\right) \end{cases}$$

And, finally:

$$\boxed{\begin{aligned} &\begin{cases} V_C(t \le 0) = V_1 \\ \\ V_C(t > 0) = V_2 + (V_1 - V_2)\exp\left(-\dfrac{t}{RC}\right) \end{cases} && (17) \\ &\begin{cases} V_R(t \le 0) \equiv 0 \\ \\ V_R(t > 0) = (V_2 - V_1)\exp\left(-\dfrac{t}{RC}\right) \end{cases} && (18) \end{aligned}}$$

Let us calculate the response on a single positive square pulse localised within the time interval $[t_1, t_2]$. The pulse amplitude is V_p and the offset is zero.

$$\begin{cases} V_C(t \le t_1) \equiv 0 \\ \\ V_C(t_1 \le t \le t_2) = \dfrac{V_p}{RC}\displaystyle\int_{t_1}^{t} \exp\left(-\dfrac{(t-s)}{RC}\right) ds \\ V_C(t_2 \le t) = \dfrac{V_p}{RC}\displaystyle\int_{t_1}^{t_2} \exp\left(-\dfrac{(t-s)}{RC}\right) ds \end{cases}$$

Calculating these integrals, we obtain:

$$\begin{cases} V_C(t \le t_1) \equiv 0 \\ \\ V_C(t_1 \le t \le t_2) = V_p\left(1 - \exp\left(\dfrac{t_1 - t}{RC}\right)\right) \\ V_C(t_2 \le t) = V_p\left(\exp\left(\dfrac{t_2 - t}{RC}\right) - \exp\left(\dfrac{t_1 - t}{RC}\right)\right) \end{cases} \qquad (19)$$

$$\begin{cases} V_R(t \le t_1) \equiv 0 \\ V_R(t_1 \le t \le t_2) = V_p \exp\left(\dfrac{t_1 - t}{RC}\right) \\ V_R(t_2 \le t) = V_p \left(\exp\left(\dfrac{t_1 - t}{RC}\right) - \exp\left(\dfrac{t_2 - t}{RC}\right) \right) \end{cases} \tag{20}$$

Using the step function $\theta(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$, Eqs. (19) and (20) can be rewritten in the following form:

$$V_C(t) = V_p \left[\theta(t - t_1)\theta(t_2 - t) - \left(\theta(t - t_1)\theta(t_2 - t) + \theta(t - t_2)\right)\exp\left(\frac{t_1 - t}{RC}\right) + \theta(t - t_2)\exp\left(\frac{t_2 - t}{RC}\right) \right] \tag{21}$$

$$V_R(t) = V_p \left[\left(\theta(t - t_1)\theta(t_2 - t) + \theta(t - t_2)\right)\exp\left(\frac{t_1 - t}{RC}\right) - \theta(t - t_2)\exp\left(\frac{t_2 - t}{RC}\right) \right] \tag{22}$$

For a periodical input parameter $V_{in}(t) = V_{in}(t + T)$, where $t \in [t_0, t_0 + T]$, t_0 is the reference time, and T is the period, the following equation can be derived:

$$V_{out}(t_0 \le t \le t_0 + T) = V_{offset} \int_{-\infty}^{t} F(t - s)ds + \int_{t_0}^{t} F(t - s)\left(V_{in}(s) - V_{offset}\right)ds + \int_{t_0}^{t_0 + T} \tilde{F}(t - s)\left(V_{in}(s) - V_{offset}\right)ds \tag{23}$$

where $\tilde{F}(t > 0) = \sum_{n=1}^{\infty} F(t + nT)$ is a periodical function, $V_{offset} = V_{in}(t_0)$, and $V_{out}(t)$ is a periodical output.

Since in many tasks $\tilde{F}(t > 0) = \sum_{n=1}^{\infty} F(t + nT)$ will be a combination of exponential functions (see an example below), the integrals in Eq. 23 can be calculated analytically for different waveforms like square pulses, saw-tooth waveform, and so on. Therefore, the final result will be obtained in a compact formula form. **This approach can be considered as an alternative to Eq. (1.10), where the output is represented in the form of Fourier series:**

$$V_{out}(t) = \hat{F}(0)\frac{a_0}{2} +$$
$$+ \sum_{k=1}^{\infty} \left[\left(a_k \operatorname{Re}\left[\hat{F}(\omega_k)\right] + b_k \operatorname{Im}\left[\hat{F}(\omega_k)\right] \right)\cos(\omega_k t) + \left(b_k \operatorname{Re}\left[\hat{F}(\omega_k)\right] - a_k \operatorname{Im}\left[\hat{F}(\omega_k)\right] \right)\sin(\omega_k t) \right]$$

To prove Eq. (23), we will start with $\int_{-\infty}^{t} F(t - s)V_{in}(s)ds$, *where* $V_{in}(t)$ *can be written in an equivalent form* $V_{in}(t) = V_{in}(t_0) + \left(V_{in}(t) - V_{in}(t_0)\right)$*:*

$$\int_{-\infty}^{t} F(t - s)V_{in}(s)ds = V_{in}(t_0) \int_{-\infty}^{t} F(t - s)ds + \int_{-\infty}^{t} F(t - s)\left(V_{in}(s) - V_{in}(t_0)\right)ds$$

In turn, the periodical function $\left(V_{in}(s)-V_{in}(t_0)\right)$ *can be represented for* $t<t_0+T$ *as the sum of the piecewise functions ("pulses")* $V_{in}^{(k)}(t)\Big|_{k\geq 1}=\begin{cases}\left(V_{in}(t)-V_{in}(t_0)\right), & t\in[t_0-(k-1)T,t_0-T(k-2)]\\ 0, & t\notin[t_0-(k-1)T,t_0-T(k-2)]\end{cases}$. *Since* $V_{out}(t)$ *is also a periodical function, it would be enough to calculate it only within* $t\in[t_0,t_0+T]$. *The first pulse* $V_{in}^{(1)}(t)=\begin{cases}\left(V_{in}(t)-V_{in}(t_0)\right), & t\in[t_0,t_0+T]\\ 0, & t\notin[t_0,t_0+T]\end{cases}$ *gives the following contribution to the total output parameter:*

$$V_{out}^{(1)}(t_0\leq t\leq t_0+T)=\int_{t_0}^{t}F(t-s)\left(V_{in}(s)-V_{in}(t_0)\right)ds\,.$$

The second pulse $V_{in}^{(2)}(t)=\begin{cases}\left(V_{in}(t)-V_{in}(t_0)\right), & t\in[t_0-T,t_0]\\ 0, & t\notin t\in[t_0-T,t_0]\end{cases}$ *is shifted back by* T *with respect to the first one and it gives:*

$$V_{out}^{(2)}(t_0\leq t\leq t_0+T)=\int_{t_0-T}^{t_0}F(t-s)\left(V_{in}(s)-V_{in}(t_0)\right)ds\Rightarrow\{q=s+T\}\Rightarrow$$

$$\Rightarrow\int_{t_0}^{t_0+T}F(t+T-q)\left(V_{in}(q-T)-V_{in}(t_0)\right)dq$$

Owing to the periodicity of $V_{in}(t)$, *we obtain:*

$$V_{out}^{(2)}(t)=\int_{t_0}^{t_0+T}F(t+T-s)\left(V_{in}(s)-V_{in}(t_0)\right)ds$$

where we designate the integration variable q *as* s. *If we continue this process, we obtain for k-th pulse:*

$$V_{out}^{(k\geq 2)}(t_0\leq t\leq t_0+T)=\int_{t_0}^{t_0+T}F(t+(k-1)T-s)\left(V_{in}(s)-V_{in}(t_0)\right)ds$$

The final response is $V_{in}(t_0)\int_{-\infty}^{t}F(t-s)ds+\int_{t_0}^{t}F(t-s)\left(V_{in}(s)-V_{in}(t_0)\right)ds+\sum_{k=2}^{\infty}V_{out}^{(k)}(t)$ *or Eq. (23).*

An example how to use Eq. 23 is considered below. For a periodical $V_{in}(t)$ and $F(t>0)=\frac{1}{RC}\exp\left(-\frac{t}{RC}\right)$, we obtain:

$$V_{offset}\int_{-\infty}^{t}F(t-s)ds=V_{offset}$$

$$V_C(t_0\leq t\leq t_0+T)=V_{offset}+\frac{1}{RC}\int_{t_0}^{t}\exp\left(-\frac{(t-s)}{RC}\right)\left(V_{in}(s)-V_{offset}\right)ds+\int_{t_0}^{t_0+T}\tilde{F}(t-s)\left(V_{in}(s)-V_{offset}\right)ds$$

where

$$\tilde{F}(t>0)=\frac{1}{RC}\sum_{n=1}^{\infty}\exp\left(-\frac{(t+nT)}{RC}\right)=\frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\sum_{n=1}^{\infty}\exp\left(-\frac{nT}{RC}\right)=\frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\times\frac{\exp\left(-\frac{T}{RC}\right)}{1-\exp\left(-\frac{T}{RC}\right)}$$

Here, we used the sum of a geometric series. Finally, we obtain:

$$\boxed{\begin{aligned}V_C(t_0\le t\le t_0+T)&=V_{offset}+\frac{1}{RC}\int_{t_0}^{t}\exp\left(-\frac{(t-s)}{RC}\right)\left(V_{in}(s)-V_{offset}\right)ds+\\&+\frac{1}{RC}\times\frac{\exp\left(-\frac{T}{RC}\right)}{1-\exp\left(-\frac{T}{RC}\right)}\int_{t_0}^{t_0+T}\exp\left(-\frac{(t-s)}{RC}\right)\left(V_{in}(s)-V_{offset}\right)ds\end{aligned}}\qquad(24)$$

Using Eq. (24) for the RC network shown in Fig. 2, we can calculate its steady-state response for the periodical square pulse excitation.

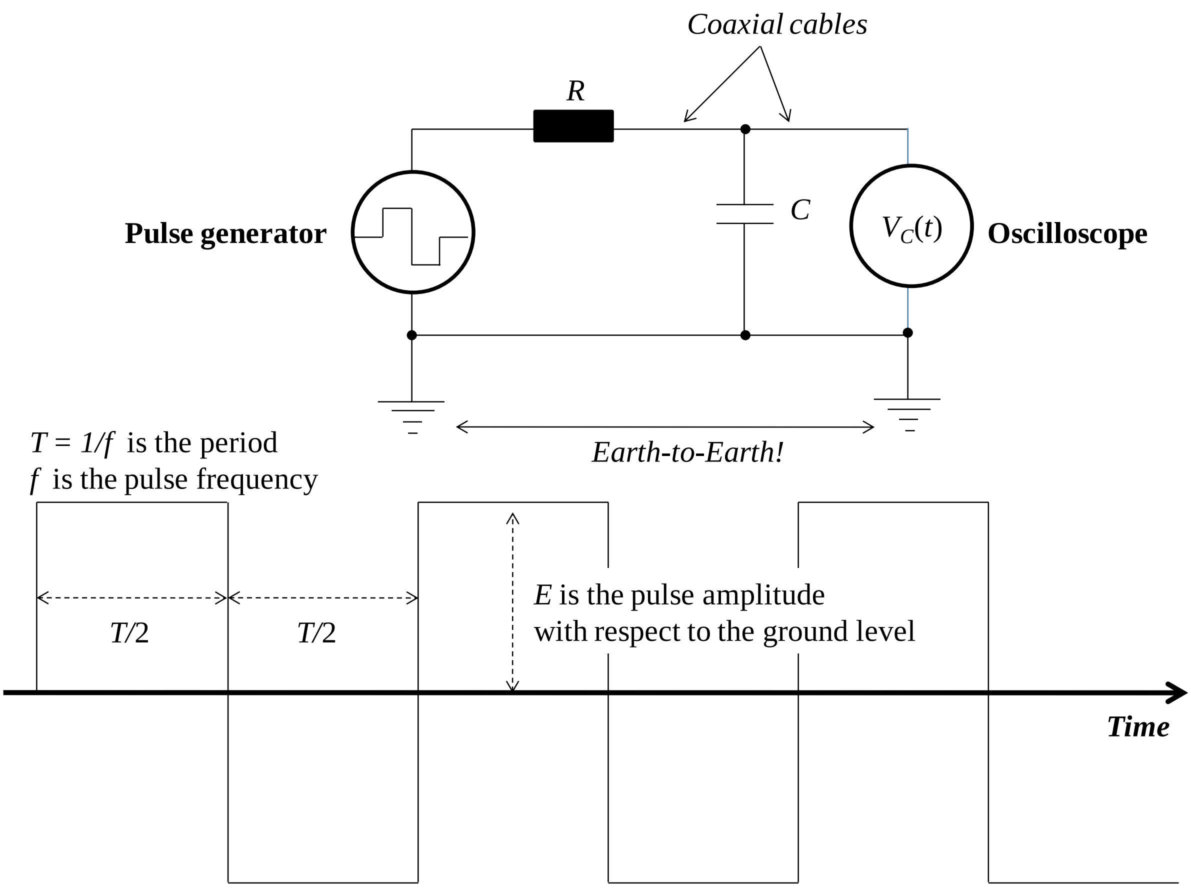


Fig. 2 RC network.

The input signal $V_{in}(t)=\begin{cases}E, & t\in]0,T/2[\\ -E, & t\in[T/2,T]\end{cases}$ is the symmetrical (±) pulse excitation. According to Eq. (23), $V_{offset}=-E$, and for $t_0=0$ we obtain:

$$V_C(0\le t\le T) = -E + \frac{1}{RC}\int_0^t \exp\left(-\frac{(t-s)}{RC}\right)(V_{in}(s)+E)ds + \\ + \frac{1}{RC}\times\frac{\exp\left(-\frac{T}{RC}\right)}{1-\exp\left(-\frac{T}{RC}\right)}\int_0^T \exp\left(-\frac{(t-s)}{RC}\right)(V_{in}(s)+E)ds \tag{25}$$

To calculate the integrals in Eq. (25), we have to divide the whole period into two intervals $[0, T/2]$ and $[T/2, T]$:

$$V_C(0\le t\le T/2) = -E + \frac{2E}{RC}\int_0^t \exp\left(-\frac{(t-s)}{RC}\right)ds + \frac{2E}{RC}\times\frac{\exp\left(-\frac{T}{RC}\right)}{1-\exp\left(-\frac{T}{RC}\right)}\int_0^{T/2} \exp\left(-\frac{(t-s)}{RC}\right)ds \tag{26}$$

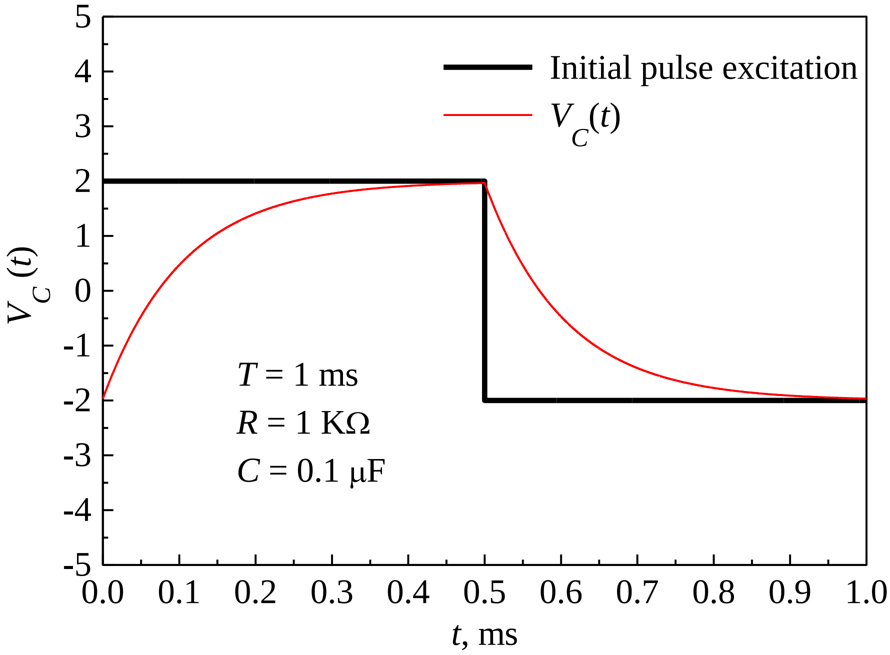
$$V_C(T/2\le t\le T) = -E + \frac{2E}{RC}\int_0^{T/2} \exp\left(-\frac{(t-s)}{RC}\right)ds + \frac{2E}{RC}\times\frac{\exp\left(-\frac{T}{RC}\right)}{1-\exp\left(-\frac{T}{RC}\right)}\int_0^{T/2} \exp\left(-\frac{(t-s)}{RC}\right)ds \tag{27}$$

Calculating the integrals in Eqs. (26) and (27), we obtain:

$$V_C(0\le t\le T/2) = E - \frac{2}{1+\exp\left(-\frac{T}{2RC}\right)}\times E\exp\left(-\frac{t}{RC}\right) \tag{28}$$

$$V_C(T/2\le t\le T) = -E + \frac{2\exp\left(\frac{T}{2RC}\right)}{1+\exp\left(-\frac{T}{2RC}\right)}\times E\exp\left(-\frac{t}{RC}\right) \tag{29}$$

The numerical calculations using Eqs. (28) and (29) are demonstrated in Fig. 3.



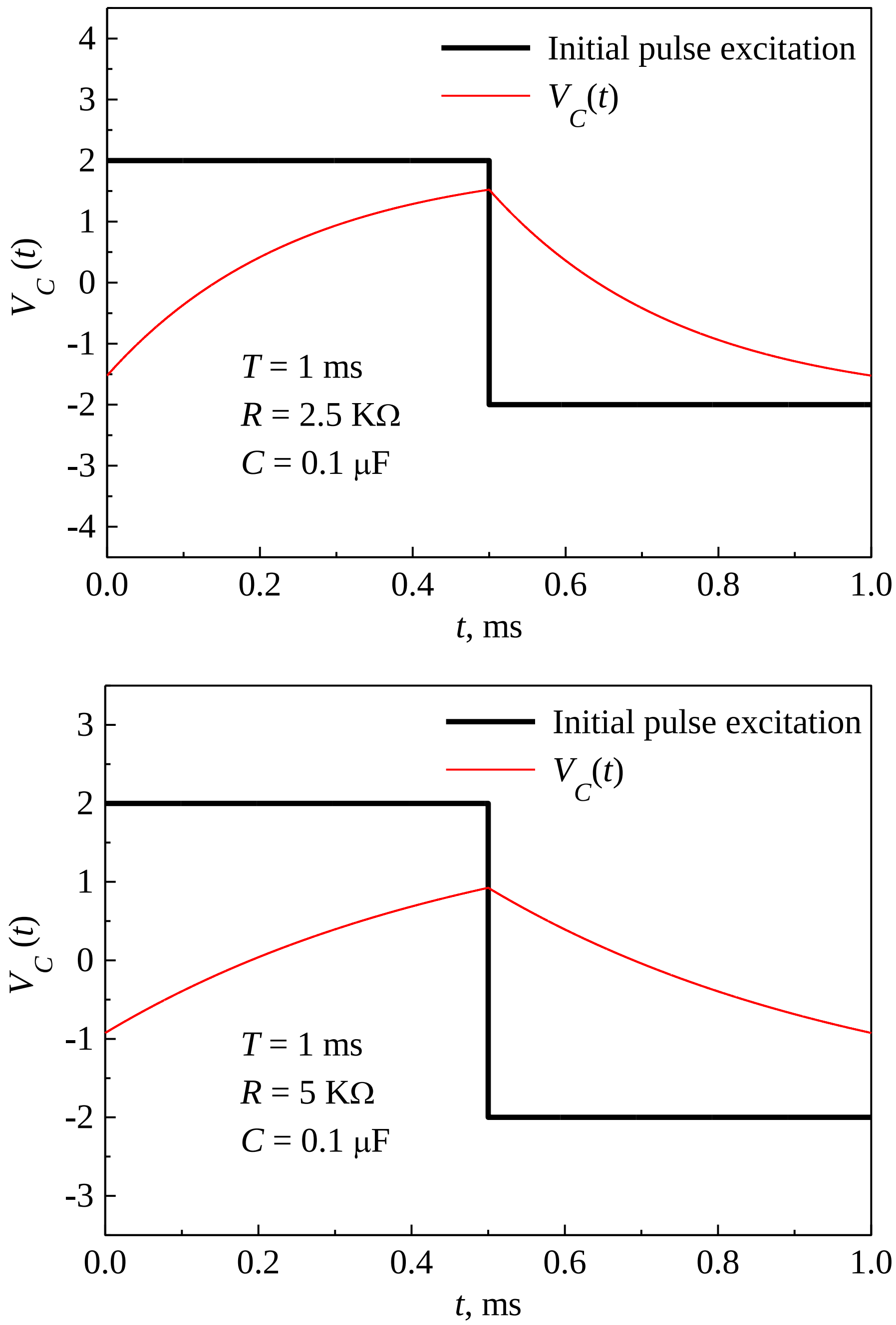


Fig. 3 Pulse response of the RC network shown in Fig. 2, where *E* = 2.

For the symmetrical triangle waveform shown in Fig. 4, there are two different methods (at least) to choose t_0 and V_{offset} in Eq. (23):

$$\begin{cases} (a)\; V_{out}(-T/4 \le t \le 3T/4) = V_{offset} \int_{-\infty}^{t} F(t-s)ds + \int_{-T/4}^{t} F(t-s)\left(V_{in}(s) - V_{offset}\right)ds + \int_{-T/4}^{3T/4} \tilde{F}(t-s)\left(V_{in}(s) - V_{offset}\right)ds \\ (b)\; V_{out}(-T/2 \le t \le T/2) = \int_{-T/2}^{t} F(t-s)V_{in}(s)ds + \int_{-T/2}^{T/2} \tilde{F}(t-s)V_{in}(s)ds \end{cases} \qquad (30)$$

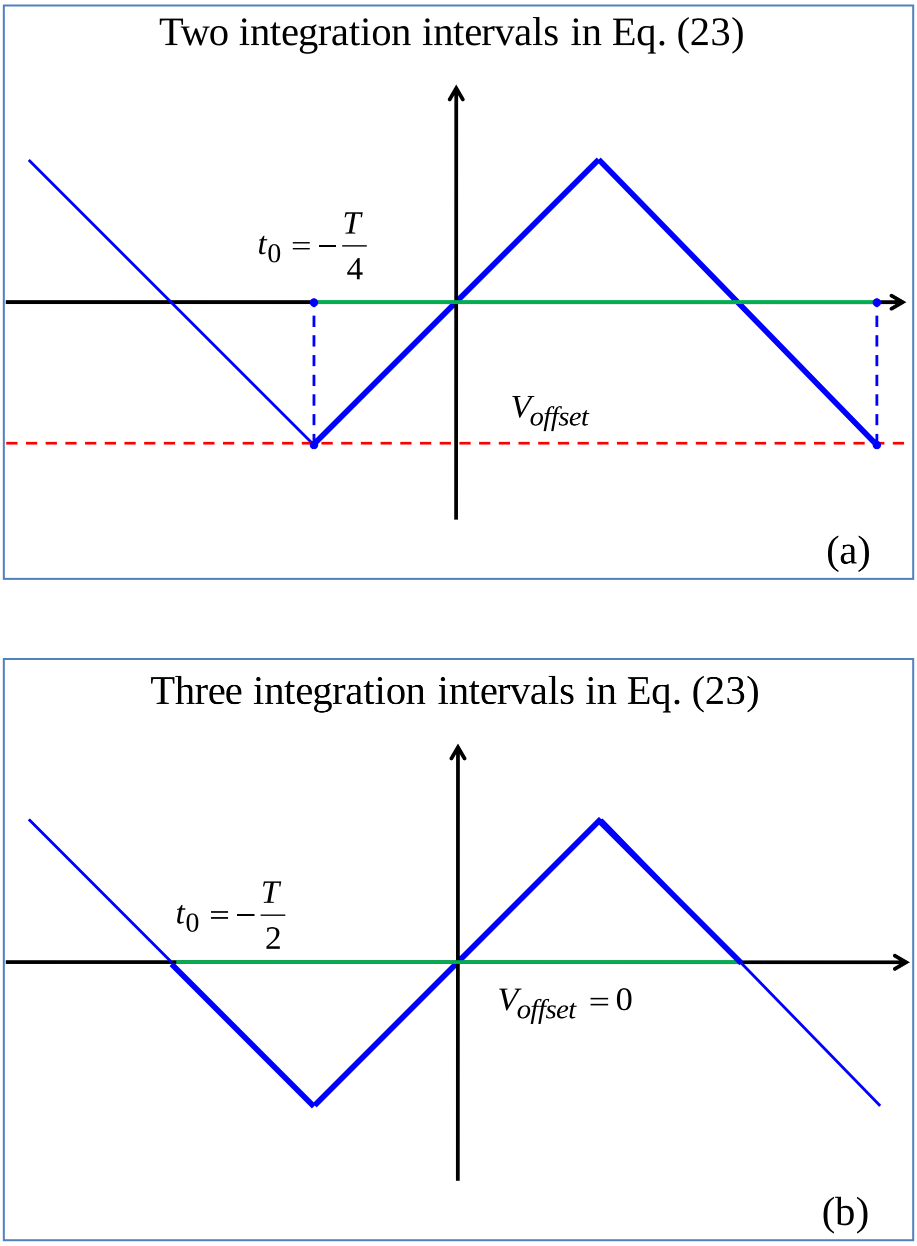


Fig. 4 Symmetrical triangle waveforms prepared for the use in Eq. (23).

III. The simplest switching voltage regulator

Transition processes in the RC (or RCL) circuits find a very important application in switching regulators. In this section, we will consider the simplest one, shown in Fig. 5. In this switching regulator, a DC input voltage E is modulated by means of the electronic switch which is driven by a pulse train with the width T_w and the pause T_p. The output signal is a stable waveform $V_C(t)$ measured across the electrolytic capacitor C and the load R_L. In Fig. 5, the input resistance R_{in} includes the internal resistance of DC source and the ON resistance of diode. Usually, $R_{in} << R_L$. In this switching regulator, we have two linear networks for two diode's conditions "”ON" and "OFF", respectively. When the diode is ON, we have the fast charge network, shown in Fig. 6(a). And, when the diode if OFF, we have the slow discharge network, shown in Fig. 6(b).

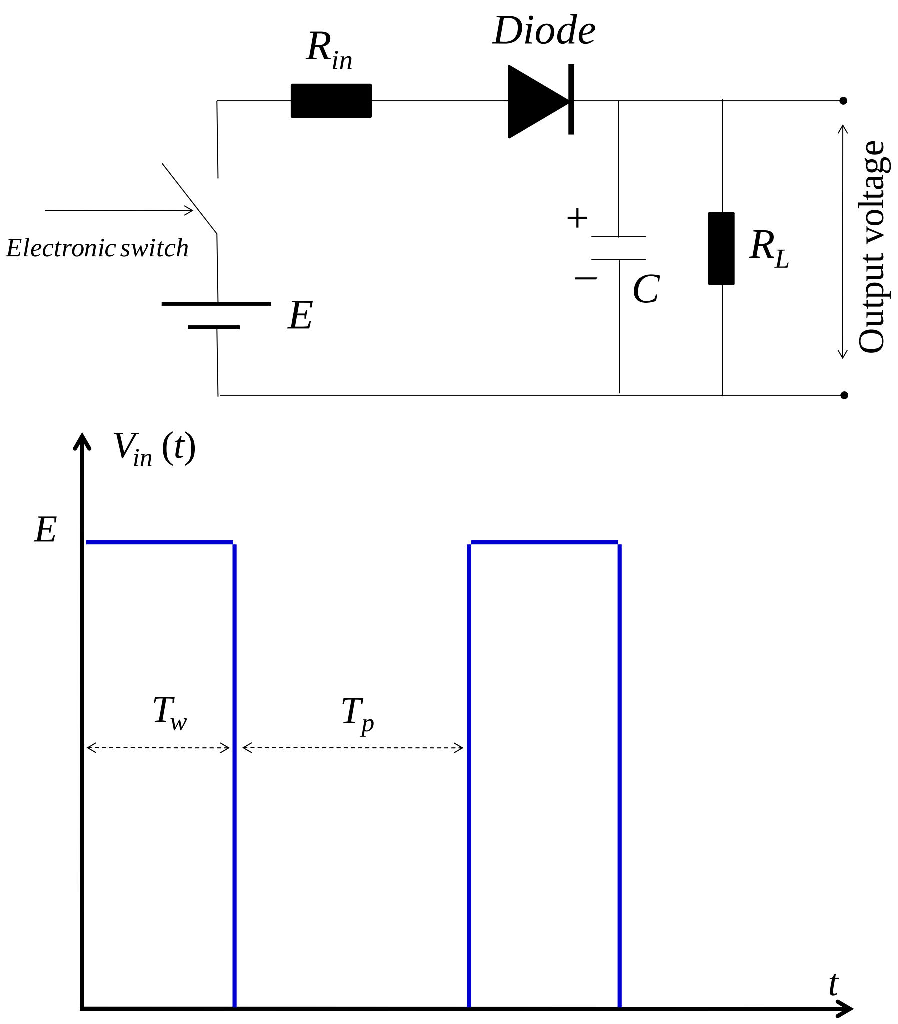


Fig. 5 Switching voltage regulator driven by a square pulse train.

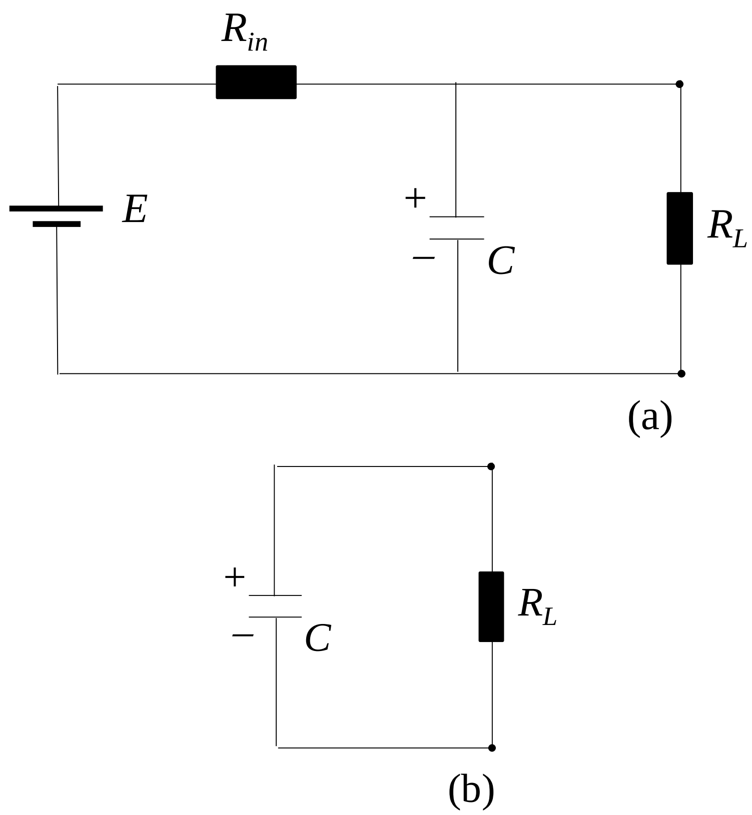


Fig. 6 Charge (a) and discharge (b) networks.

The charge and discharge networks are described by the following equations, which can be obtained from Eq. (17):

$$V_C(t) = V_0 \times \exp\left(-\frac{t}{\tau_{in}}\right) + \frac{E \times R_L}{R_{in} + R_L} \times \left(1 - \exp\left(-\frac{t}{\tau_{in}}\right)\right) \text{ – charge network} \tag{31}$$

$$V_C(t) = V_0 \times \exp\left(-\frac{t}{\tau_L}\right) \text{ – discharge network} \tag{32}$$

where $V_0 = V_C(0)$ is the initial voltage across the capacitor (or load), $\tau_{in} = C \times \left(\frac{R_{in} \times R_L}{R_{in} + R_L}\right)$ is the charge characteristic time, $\tau_L = C \times R_L$ is the discharge characteristic time, and E is the pulse amplitude (positive).

We can calculate the stable waveform amplitudes V_{C1} and V_{C2}, shown in the Fig. 7, using the periodicity condition:

$$\begin{cases} V_{C1} = V_{C2} \times \exp\left(-\frac{T_p}{\tau_L}\right) \\ V_{C2} = V_{C1} \times \exp\left(-\frac{T_w}{\tau_{in}}\right) + \frac{E \times R_L}{R_{in} + R_L} \times \left(1 - \exp\left(-\frac{T_w}{\tau_{in}}\right)\right) \end{cases} \tag{33}$$

Solving this system of linear equations, we obtain:

$$V_{C1} = \frac{E \times R_L}{R_{in} + R_L} \times \frac{\exp\left(-\frac{T_p}{\tau_L}\right) - \exp\left(-\frac{T_p \times \tau_{in} + T_w \times \tau_L}{\tau_{in} \times \tau_L}\right)}{1 - \exp\left(-\frac{T_p \times \tau_{in} + T_w \times \tau_L}{\tau_{in} \times \tau_L}\right)} \tag{34}$$

$$V_{C2} = \frac{E \times R_L}{R_{in} + R_L} \times \frac{1 - \exp\left(-\frac{T_w}{\tau_{in}}\right)}{1 - \exp\left(-\frac{T_p \times \tau_{in} + T_w \times \tau_L}{\tau_{in} \times \tau_L}\right)} \tag{35}$$

$$\text{Ripple} = V_{C2} - V_{C1} = \frac{E \times R_L}{R_{in} + R_L} \times \frac{\left(1 - \exp\left(-\frac{T_p}{\tau_L}\right)\right) \times \left(1 - \exp\left(-\frac{T_w}{\tau_{in}}\right)\right)}{1 - \exp\left(-\frac{T_p \times \tau_{in} + T_w \times \tau_L}{\tau_{in} \times \tau_L}\right)} \tag{36}$$

$$\text{Average} = \frac{V_{C1} + V_{C2}}{2} = \frac{E \times R_L}{2 \times (R_{in} + R_L)} \times \frac{\left(1 + \exp\left(-\frac{T_p}{\tau_L}\right)\right) \times \left(1 - \exp\left(-\frac{T_w}{\tau_{in}}\right)\right)}{1 - \exp\left(-\frac{T_p \times \tau_{in} + T_w \times \tau_L}{\tau_{in} \times \tau_L}\right)} \tag{37}$$

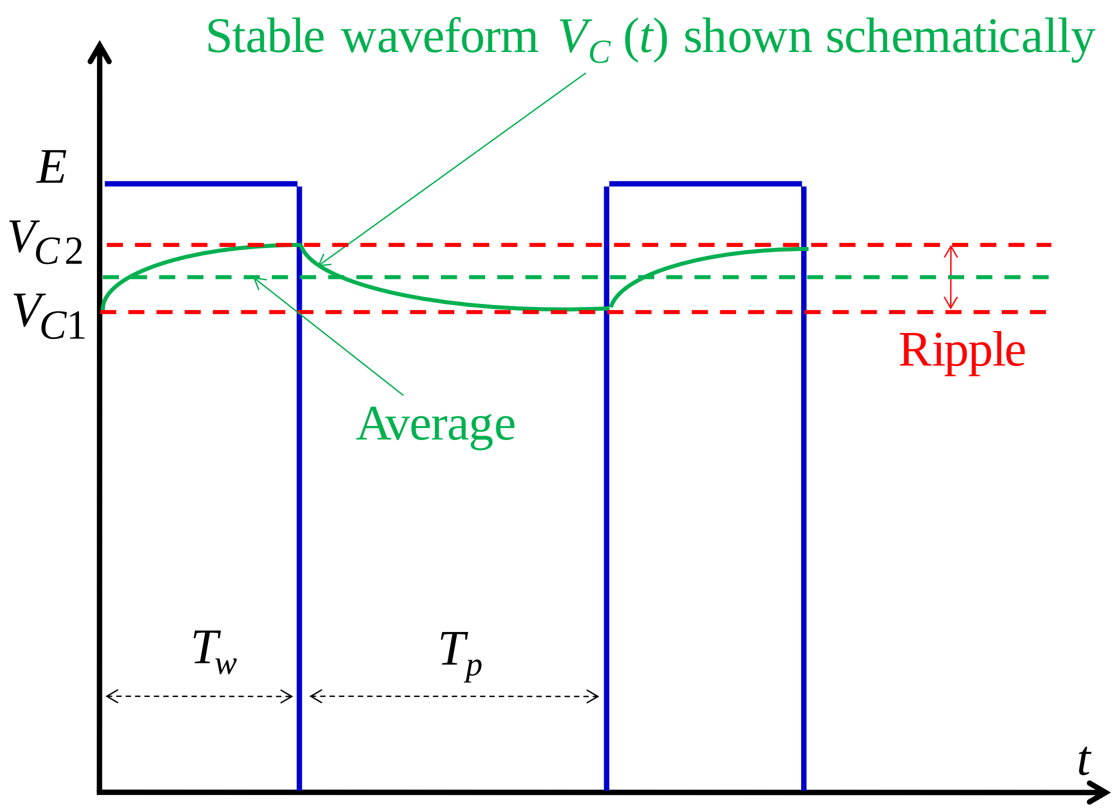


Fig. 7 Stable output waveform measured across the capacitor.

Now we are able to reproduce the stable waveform $V_C(t)$ – a periodical piecewise continuous function:

$$V_C(t) = V_{C2} \times \exp\left(-\frac{t}{\tau_L}\right) \text{ – for the pause intervals} \tag{38}$$

$$V_C(t) = V_{C1} \times \exp\left(-\frac{t}{\tau_{in}}\right) + \frac{E \times R_L}{R_{in} + R_L} \times \left(1 - \exp\left(-\frac{t}{\tau_{in}}\right)\right) \text{ – for the positive pulse intervals} \tag{39}$$

where V_{C1} and V_{C2} are the amplitudes from Eqs. (34) and (35). The whole periodical function $V_C(t)$ within the period $T = T_p + T_w$ can be written in the following equivalent forms:

$$V_C(t > 0) = \begin{cases} V_{C2} \times \exp\left(-\dfrac{t}{\tau_L}\right), & t \in [0, T_p] \\ V_{C1} \times \exp\left(-\dfrac{(t - T_p)}{\tau_{in}}\right) + \dfrac{E \times R_L}{R_{in} + R_L} \times \left(1 - \exp\left(-\dfrac{(t - T_p)}{\tau_{in}}\right)\right), & t \in [T_p, (T_p + T_w)] \end{cases} \tag{40}$$

OR

$$V_C(t > 0) = \begin{cases} V_{C1} \times \exp\left(-\dfrac{t}{\tau_{in}}\right) + \dfrac{E \times R_L}{R_{in} + R_L} \times \left(1 - \exp\left(-\dfrac{t}{\tau_{in}}\right)\right), & t \in [0, T_w] \\ V_{C2} \times \exp\left(-\dfrac{(t - T_w)}{\tau_L}\right), & t \in [T_w, (T_w + T_p)] \end{cases} \tag{41}$$

IV. Ideal operational amplifier as a linear network

An operational amplifier ("op amp") is a differential input, single ended output amplifier, as shown symbolically in Fig. 8. This device is an amplifier intended for use with external feedback elements, where these elements determine the resultant function, or operation. This gives rise to the name "operational amplifier". At this point, note that there is no need for concern with any actual technology to implement the amplifier. Attention is focused more on the behavioral nature of this building block device.

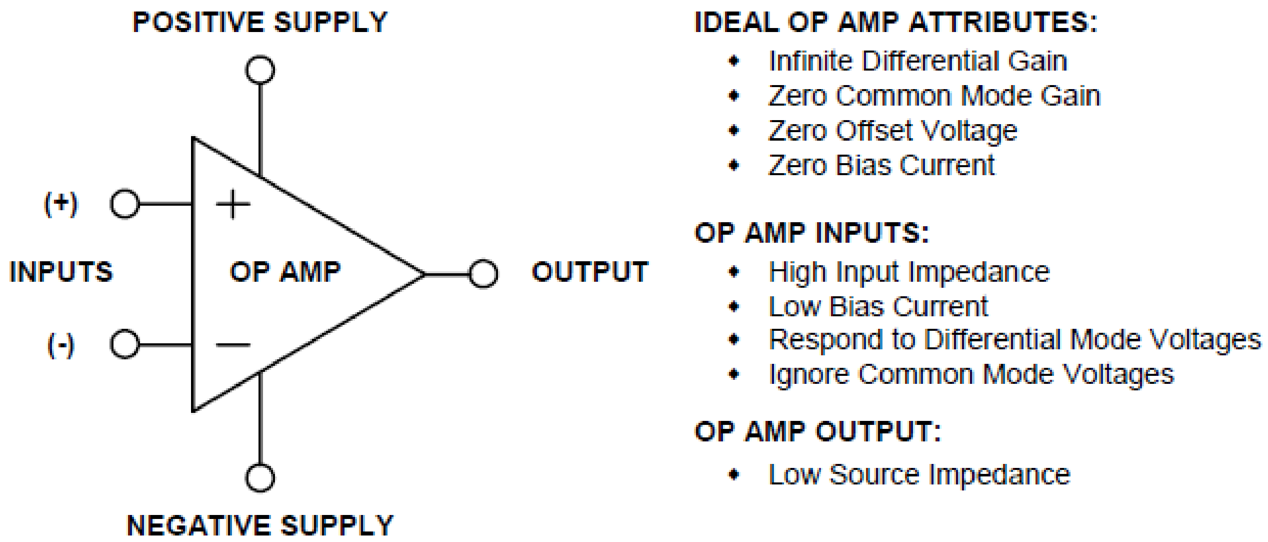


Fig. 8 The ideal op amp and its attributes.

We will consider an op amp as a linear network between its differential input signals and the output. The transfer function of this network has the special name "gain". An ideal op amp has infinite gain for differential input signals. In practice, real devices will have quite high gain (also called open-loop gain) but this gain would not necessary be precisely known. In terms of specifications, gain is measured in terms of $\frac{V_{out}}{V_{in}}$ and it is dimensionless. Here, V_{in} is the differential input voltage and V_{out} is the output voltage.

Since in practice the gain depends on the frequency, it must be understood as the ratio $\hat{A}(\omega) = \frac{\hat{V}_{out}(\omega)}{\hat{V}_{in}(\omega)}$, where $\hat{V}_{out}(\omega)$ and $\hat{V}_{in}(\omega)$ are the complex amplitude for the corresponding harmonic signals. Also, an ideal op amp has zero gain for signals common to both inputs, that is, common mode signals. The ideal op amp also has zero offset voltage, and draws zero bias current at both inputs. In practice, another important attribute is the concept of low source impedance at the output.

The basic op amp hookup of Fig. 9 applies a signal to the "+" input terminal, and a network delivers a fraction of the output voltage to the "–" input terminal. This constitutes feedback, with the op amp operating in closed loop mode. The feedback network can be resistive or reactive, linear or non-linear, or any combination of these. For a linear feedback network, we will introduce the gain $\hat{\beta}(\omega)$. In general, the op amp gain depends not only on the frequency, but also on the amplitude of signals applied to the "±" input terminals. This effect is non-linear, and hence it will result in some frequency distortions. When a sinusoidal wave suffers non-linear distortions in an amplifier, the amplifier is in effect adding harmonics to the original waveform. The negative feedback will significantly suppress these non-linear distortions.

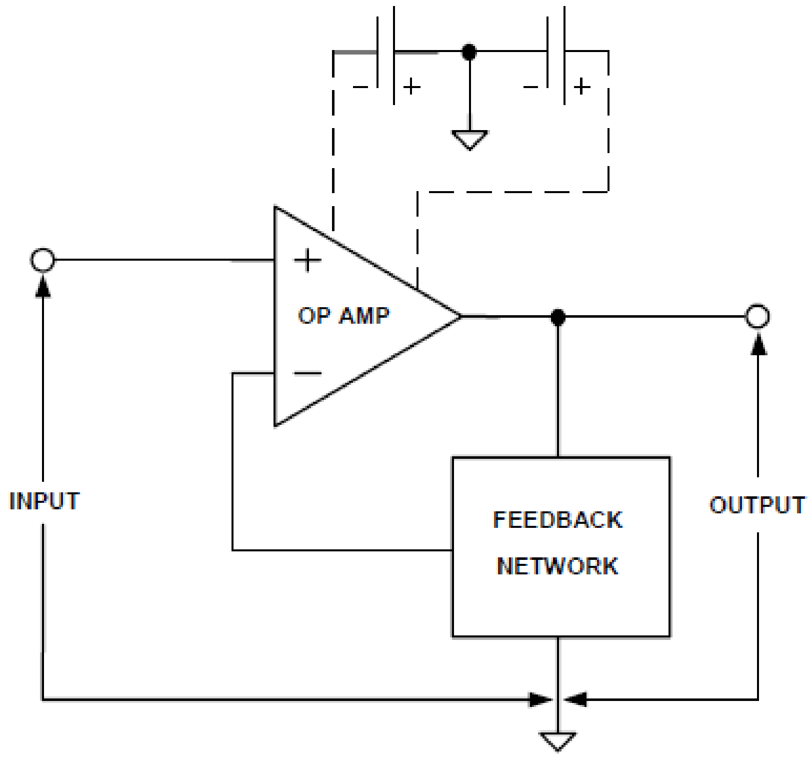


Fig. 9 A generalised op amp circuit with feedback applied.

The concept of feedback is both an essential and salient point concerning op amp use. With feedback, the net closed-loop gain characteristics become primarily dependent upon a set of external components (usually passive). Thus behavior is less dependent upon the relatively unstable amplifier open-loop characteristics. Note that in Fig. 9, the input signal is applied between the op amp "+" input and a common or reference point, as denoted by the ground symbol. It is important to note that this reference point is also common to the output and feedback network. By definition, the op amp stage's output signal appears between the output terminal/feedback network input, and this common ground. The emphasize how the input/output signals are referenced to the power supply, dual supply connections are shown dotted, with the "±" power supply midpoint common to the input/output signal ground. But do note, while all op amp application circuits may not show full details of the power supply connections, every real circuit will always use power supplies.

The negative feedback reduces non-linear distortions by the same factor as it reduces gain (prove this equation!):

$$\hat{A}_f(\omega) = \frac{\hat{A}(\omega)}{1+\hat{\beta}(\omega)\hat{A}(\omega)}, \quad (42)$$

where $\hat{A}(\omega)$ is the open-loop gain, $\hat{A}_f(\omega)$ is the gain with a feedback, and $|1+\hat{\beta}(\omega)\hat{A}(\omega)|>1$ is the condition of "negative feedback". If $|1+\hat{\beta}(\omega)\hat{A}(\omega)|<1$, the feedback is termed "positive", or "regenerative". For a positive feedback, $|\hat{A}_f(\omega)|$ will be greater than $|\hat{A}(\omega)|$. Because of the reduced stability of an amplifier with positive feedback, it is seldom used. For the amplifier stability, all the poles of $\hat{A}_f(\omega)$ must be located in the complex upper half plane. Although negative feedback appears to be the panacea for all amplifier ailments it is important to note that it is only effective as long as the open-loop gain $\hat{A}(\omega)$ remains much greater than the close-loop gain $\hat{A}_f(\omega)$. In this case, we obtain from Eq. (42):

$$\boxed{\hat{A}_f(\omega) \approx \frac{1}{\hat{\beta}(\omega)} \qquad (43)}$$

If the negative feedback is given too strong, the rise time (time for waveform to rise from 0.1 to 0.9 of its steady-state value) is greatly decreased, but this improvement is obtained at the expense of **a ringing (oscillatory) response** that is unacceptable for many applications.

For a perfectly balanced op amplifier, the output signal $V_{out}(t)$ can be calculated as:

$$V_{out}(t) = \int_{-\infty}^{t} A_f(t-s) V_{in}(s) ds \qquad (44)$$

where

$$A_f(t>0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{A}_f(\omega) \exp(i\omega t) d\omega = \frac{1}{\pi} \int_{0}^{+\infty} \left(\mathrm{Re}\left[\hat{A}_f(\omega)\right] \cos(\omega t) - \mathrm{Im}\left[\hat{A}_f(\omega)\right] \sin(\omega t) \right) d\omega \qquad (45)$$

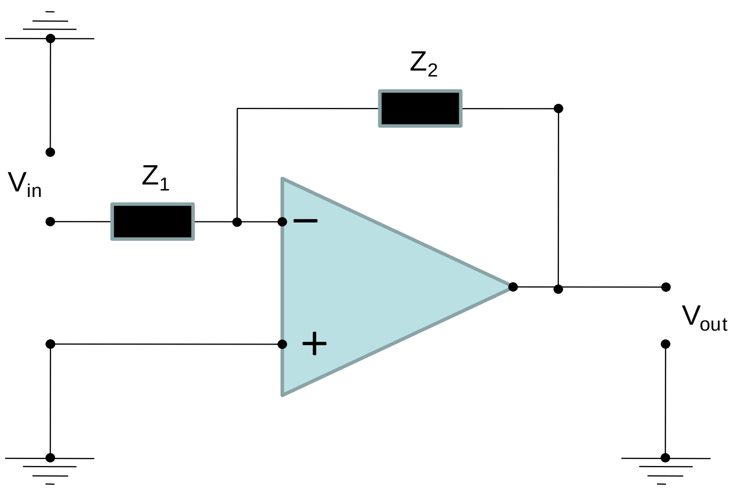
and $V_{in}(t)$ is the input signal between an input terminal ("+" or "–") and the reference point (ground). According to Eq. (1.10), for a periodical input signal $V_{in}(t) = V_{in}(t+T)$, we have:

$$V_{out}(t) = \hat{A}_f(0) \frac{a_0}{2} + \\ + \sum_{k=1}^{\infty} \left[\left(a_k \,\mathrm{Re}\left[\hat{A}_f(\omega_k)\right] + b_k \,\mathrm{Im}\left[\hat{A}_f(\omega_k)\right] \right) \cos(\omega_k t) + \left(b_k \,\mathrm{Re}\left[\hat{A}_f(\omega_k)\right] - a_k \,\mathrm{Im}\left[\hat{A}_f(\omega_k)\right] \right) \sin(\omega_k t) \right] \qquad (46)$$

In Eq. (46), the Fourier series of $V_{in}(t)$ is used.

Using operational amplifiers, we can engineer the different transfer functions. The gain $\hat{A}(\omega)$ of the ideal open-loop operational amplifier is infinite. With the voltage-shunt feedback, the gain $\hat{A}_f(\omega)$ is defined by the certain impedance ratios, as shown in Eqs. (47) and (48).

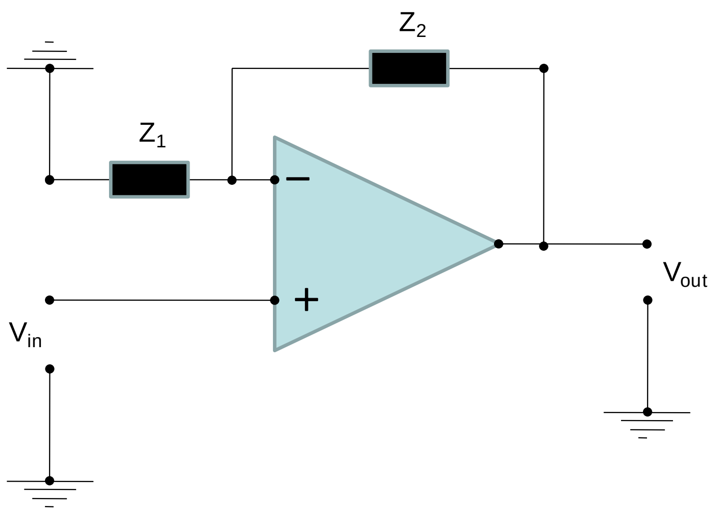
Inverting operational amplifier with the voltage-shunt feedback



$$\hat{A}_f^-(\omega) = \frac{\hat{V}_{out}(\omega)}{\hat{V}_{in}(\omega)} = -\frac{Z_2}{Z_1} \quad \textbf{(Inverting OpAmp)} \tag{47}$$

Here, Z_1 and Z_2 are any impedances.

Non-inverting operational amplifier with the voltage-shunt feedback



$$\hat{A}_f^+(\omega) = \frac{\hat{V}_{out}(\omega)}{\hat{V}_{in}(\omega)} = 1 + \frac{Z_2}{Z_1} \quad \textbf{(Non-inverting OpAmp)} \tag{48}$$

Here, Z_1 and Z_2 are any impedances.

Some simplest examples:

a) Inverting integrator, where $Z_1 = R$ **(resistor) and** $Z_2 = -\frac{i}{C\omega}$ **(capacitor).**

$\hat{A}_f^-(\omega) = -\frac{Z_2}{Z_1} = \frac{i}{RC\omega}$ is the gain (transfer function) of this circuit. Since $\lim_{\omega\to\infty} \hat{A}_f^-(\omega) = 0$, there is no growing or constant part in this gain. The function $\hat{A}_f^-(\omega) = \frac{i}{RC\omega}$ has only one pole $\omega_1 = 0$, and the corresponding residue is $\frac{i}{RC}$. Using Eq. (4.53), we obtain:

$$A_f^-(t) = ires[\hat{A}_{Vf}^-(\omega_1)]\exp(i\omega_1 t) = -\frac{1}{RC} \tag{49}$$

Using Eq. (44), we obtain:

$$\boxed{V_{out}(t) = \int_{-\infty}^{t} A_f^-(t-s)V_{in}(s)ds = -\frac{1}{RC}\int_{-\infty}^{t} V_{in}(s)ds} \tag{50}$$

b) Inverting differentiator, where $Z_1 = -\frac{i}{C\omega}$ (capacitor) and $Z_2 = R$ (resistor).

$\hat{A}_f^-(\omega) = -\frac{Z_2}{Z_1} = -iRC\omega$ is the gain (transfer function) of this circuit. The function $\hat{A}_f^-(\omega) = -iRC\omega$ does not have any poles, but it is a first order polynomial with $A_1 = -iRC$ (see (61)). Using Eq. (4.53), we obtain:

$$\hat{A}_f^-(t) = -iA_1\frac{d\delta(t)}{dt} = -RC\frac{d\delta(t)}{dt} \tag{51}$$

Using Eqs. (44) and (4.37), we obtain:

$$\boxed{V_{out}(t) = \int_{-\infty}^{t} A_f^-(t-s)V_{in}(s)ds = -RC\int_{-\infty}^{t} \frac{d\delta(t-s)}{ds}V_{in}(s)ds = -RC\frac{dV_{in}(t)}{dt}} \tag{52}$$

c) Non-inverting integrator, where $Z_1 = R$ (resistor) and $Z_2 = -\frac{i}{C\omega}$ (capacitor).

$\hat{A}_f^+(\omega) = 1 + \frac{Z_2}{Z_1} = 1 - \frac{i}{RC\omega}$ is the gain (transfer function) of this circuit. Since $\lim_{\omega\to\infty} \hat{A}_f^+(\omega) = 1$, there is a constant part $A_0 = 1$ in this gain. The function $\hat{A}_f^+(\omega) = 1 - \frac{i}{RC\omega}$ has only one pole $\omega_1 = 0$, and the corresponding residue is $-\frac{i}{RC}$. Using Eq. (4.53), we obtain:

$$A_f^+(t) = A_0\delta(t) + i\left(-\frac{i}{RC}\right) = \delta(t) + \frac{1}{RC} \tag{53}$$

Using Eq. (44), we obtain:

$$\boxed{V_{out}(t) = \int_{-\infty}^{t} A_f^+(t-s)V_{in}(s)ds = \int_{-\infty}^{t}\left(\delta(t-s) + \frac{1}{RC}\right)V_{in}(s)ds = V_{in}(t) + \frac{1}{RC}\int_{-\infty}^{t} V_{in}(s)ds} \tag{54}$$

d) Non-inverting differentiator, where $Z_1 = -\dfrac{i}{C\omega}$ (capacitor) and $Z_2 = R$ (resistor).

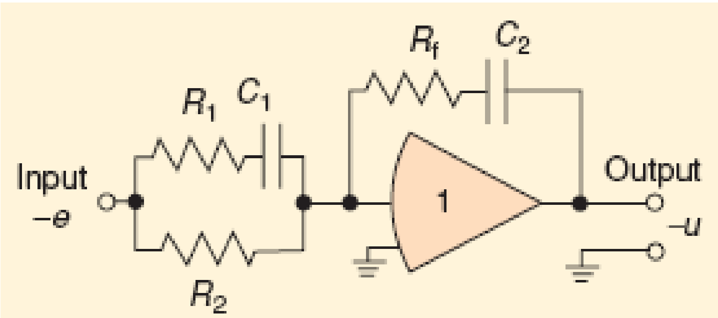
$\hat{A}_f^+(\omega) = 1 + \dfrac{Z_2}{Z_1} = 1 + iRC\omega$ is the gain (transfer function) of this circuit. The function $\hat{A}_f^+(\omega) = 1 + iRC\omega$ does not have any poles, but it is a first order polynomial with $A_0 = 1$ and $A_1 = iRC$. Using Eq. (4.53), we obtain:

$$\hat{A}_f^+(t) = A_0\delta(t) - iA_1\frac{d\delta(t)}{dt} = \delta(t) + RC\frac{d\delta(t)}{dt} \qquad (55)$$

Using Eqs. (44) and (4.37), we obtain:

$$\boxed{V_{out}(t) = \int_{-\infty}^{t} A_f^+(t-s)V_{in}(s)ds = \int_{-\infty}^{t}\left(\delta(t-s) + RC\frac{d\delta(t-s)}{ds}\right)V_{in}(s)ds = V_{in}(t) + RC\frac{dV_{in}(t)}{dt}} \qquad (56)$$

A more complicated example: calculate $\hat{A}_f^-(\omega) = -\dfrac{Z_2}{Z_1}$ and $A_f^-(t>0) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{A}_f^-(\omega)\exp(i\omega t)d\omega$, for the circuit shown in the Figure below (**from a review paper on analog computations**). Then, express $V_{out}(t)$ through the convolution of $A_f^-(t>0)$ and $V_{in}(t)$.



$$Z_1 = \frac{R_2\left(R_1 - \dfrac{i}{C_1\omega}\right)}{R_1 + R_2 - \dfrac{i}{C_1\omega}} = \frac{R_2(R_1C_1\omega - i)}{(R_1+R_2)C_1\omega - i} \qquad (57)$$

$$Z_2 = R_f - \frac{i}{C_2\omega} = \frac{R_fC_2\omega - i}{C_2\omega} \qquad (R_f \text{ – feedback resistor}) \qquad (58)$$

$$\boxed{\begin{aligned}\hat{A}_f^-(\omega) &= -\frac{Z_2}{Z_1} = -\frac{(R_fC_2\omega - i)((R_1+R_2)C_1\omega - i)}{R_2C_2\omega(R_1C_1\omega - i)} = \\ &= -\frac{R_f(R_1+R_2)}{R_1R_2} + \frac{R_1 + i\omega\left(R_1(R_1+R_2)C_1 - R_fR_2C_2\right)}{R_1^2R_2C_1C_2\omega\left(\omega - \dfrac{i}{R_1C_1}\right)}\end{aligned}} \qquad (59)$$

In Eq. (59), we have extracted the constant part $\left[-\frac{R_f(R_1+R_2)}{R_1R_2}\right]$ $(\omega \to \infty)$. Then, $\omega_1 = 0$ and $\omega_2 = \frac{i}{R_1C_1}$ are the poles of $\hat{A}_{f0}^{-}(z) = \frac{R_1 + iz\left(R_1(R_1+R_2)C_1 - R_fR_2C_2\right)}{R_1^2R_2C_1C_2z\left(z - \frac{i}{R_1C_1}\right)}$, and $res[\hat{A}_{f0}^{-}(\omega_1)] = \frac{i}{R_2C_2}$ and $res[\hat{A}_{f0}^{-}(\omega_2)] = \frac{i(R_1C_1 - R_fC_2)}{R_1^2C_1C_2}$ are the residues of $\hat{A}_{f0}^{-}(z)$ in the poles ω_1 and ω_2, respectively. Using Eqs. (59) and (4.53), we obtain:

$$A_f^{-}(t>0) = -\frac{R_f(R_1+R_2)}{R_1R_2}\delta(t) - \frac{1}{R_2C_2} - \frac{(R_1C_1 - R_fC_2)}{R_1^2C_1C_2}\exp\left(-\frac{t}{R_1C_1}\right) \tag{60}$$

Putting Eq. (60) into Eq. (44), we obtain:

$$\begin{aligned} V_{out}(t) = \int_{-\infty}^{t} A_f^{-}(t-s)V_{in}(s)ds = -\frac{R_f(R_1+R_2)}{R_1R_2}V_{in}(t) - \frac{1}{R_2C_2}\int_{-\infty}^{t} V_{in}(s)ds - \\ -\frac{(R_1C_1 - R_fC_2)}{R_1^2C_1C_2}\int_{-\infty}^{t}\exp\left(-\frac{(t-s)}{R_1C_1}\right)V_{in}(s)ds \end{aligned} \tag{61}$$