Chapter 4
Transients

*The time-varying currents and voltages resulting from the sudden application of sources, usually due to switching, are called **transients**.*

Goal

1. First-order *RC* or *RL* Circuits.

2. Concepts of Transient Response and Steady-State Response.

3. Transient Response of First-Order Circuits to Time Constant.

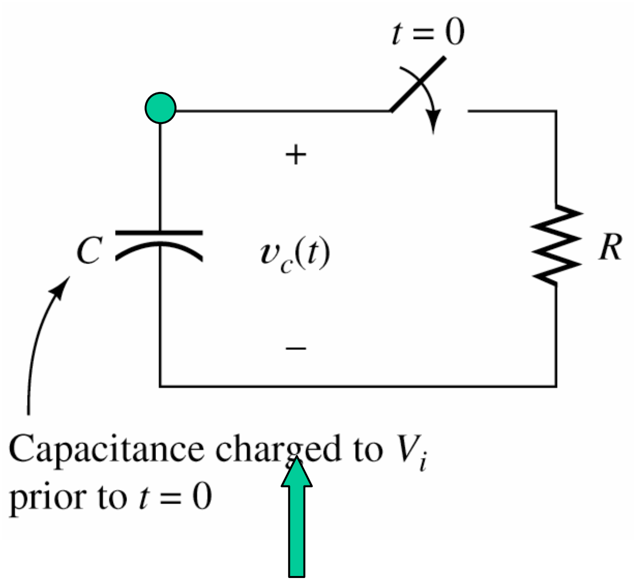
4. *RLC* Circuits in DC Steady-State Conditions.

5. Second-Order Circuits.

6. Step Response of a Second-Order System
   to its Natural Frequency and Damping Ratio.

First-Order RC Circuits

Discharge of a Capacitance through a Resistance



$$v_C(0+) = V_i$$

KCL at Node $C\dfrac{dv_C(t)}{dt} + \dfrac{v_C(t)}{R} = 0$

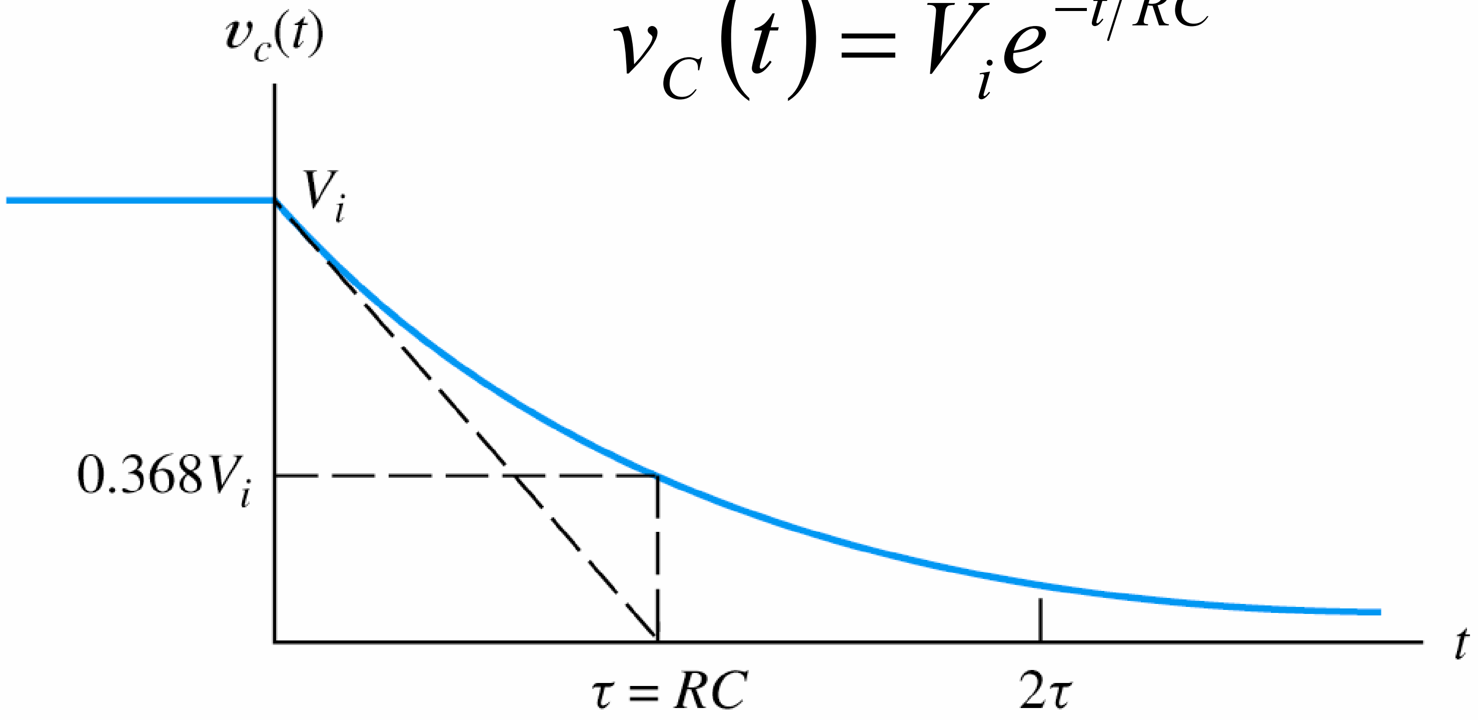
$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0 \Rightarrow v_C(t) = Ke^{st}$$

$$RCKse^{st} + Ke^{st} = 0 \Rightarrow s = \frac{-1}{RC}$$

$$v_C(t) = Ke^{-t/RC}$$

$$v_C(t) = V_i e^{-t/RC}$$

$$v_C(t) = V_i e^{-t/RC}$$



The time interval $\tau = RC$ is called the time constant of the circuit

In one time Constant, Voltage decays by factor $1/e$

Charging Capacitance from DC Source through a Resistance

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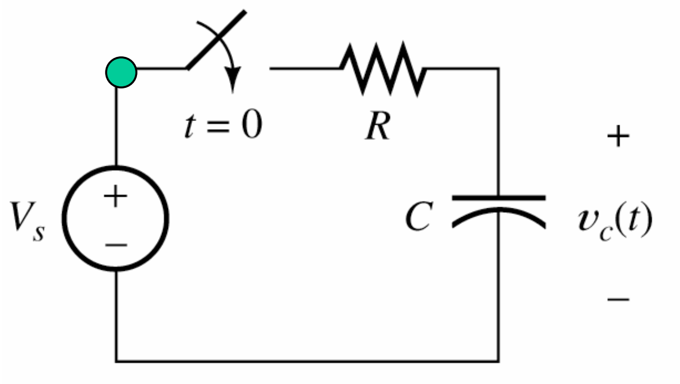

KCL at Node $C\dfrac{dv_C(t)}{dt} + \dfrac{v_C(t) - v_s}{R} = 0$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_s \Longrightarrow v_C(t) = K_1 + K_2 e^{st}$$

I.C. $v_C(0+) = v_C(0-) = 0$

$$(1 + RCs)K_2 e^{st} + K_1 = v_s \Longrightarrow s = \frac{-1}{RC} \Longrightarrow K_1 = v_s$$

$$v_C(t) = v_s + K_2 e^{-t/RC} \Longrightarrow K_2 = -v_s \qquad \tau = RC$$

$$v_C(t) = v_s - v_s e^{-t/\tau}$$

v_s : Steady State Response (Forces Response)

$v_s e^{-t/\tau}$: Transient Response

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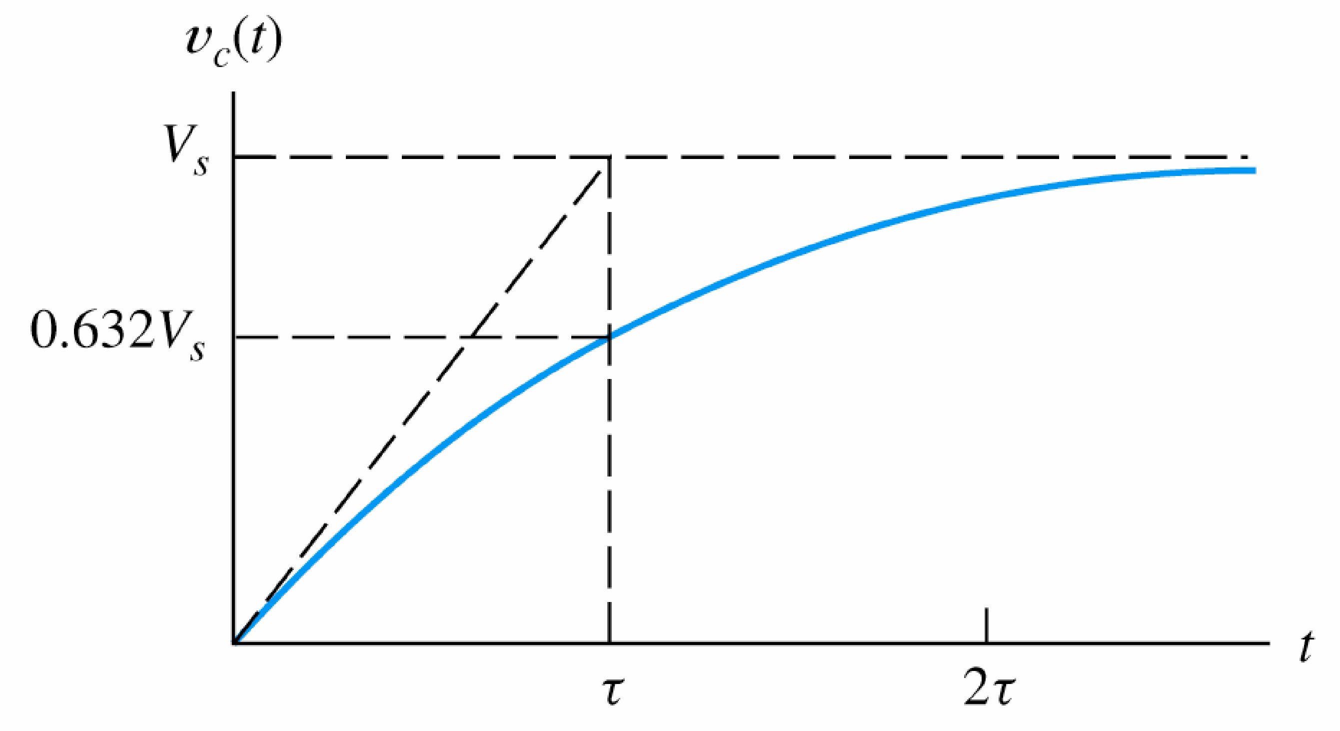
$$v_C(t) = v_s - v_s e^{-t/\tau}$$



DC STEADY STATE

At Steady State of DC Circuit,

- Voltage is Constant $i_c(t) = C\dfrac{dv_C(t)}{dt} = 0$

- Current is Constant $v_L(t) = L\dfrac{di_L(t)}{dt} = 0$

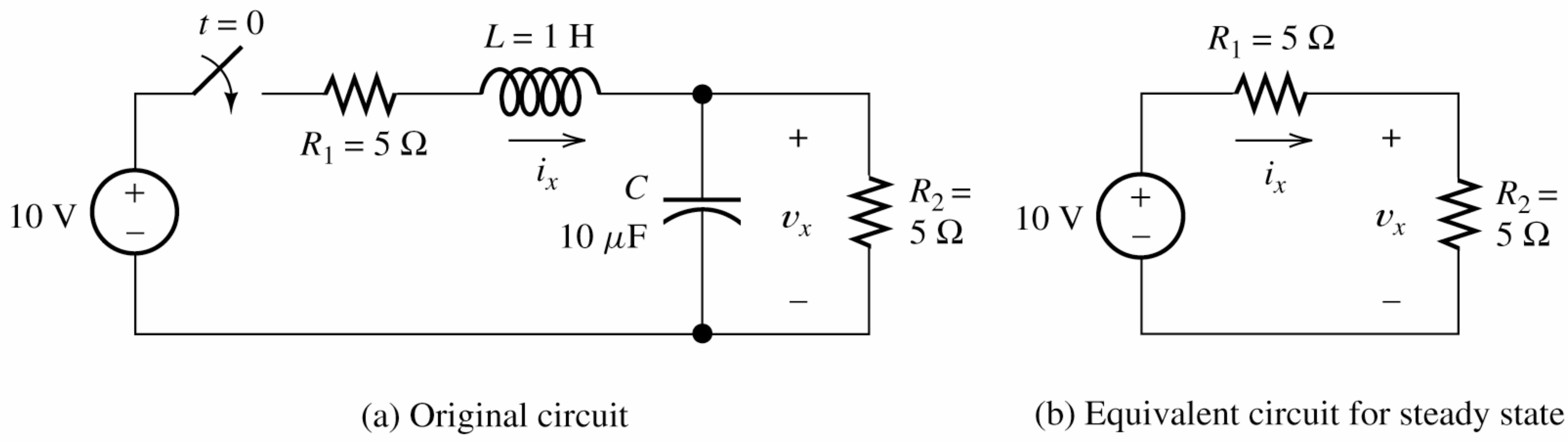
Steady State (Forced) Response for *RLC* circuits with DC sources are:

1. Replace Capacitances with open circuits.

2. Replace Inductances with short circuits.

3. Solve the remaining circuit.

Example of DC STEADY STATE



If Time is Sufficiently Large, Steady State becomes.

$$i_x = \frac{v_s}{R_1 + R_2} = \frac{10}{5+5} = 1A$$

$$v_x = R_2 i_x = 5V$$

RL Circuits

To solve circuits with DC sources, Resistances, & One energy-storage element (Inductance or Capacitance) are:

1. Apply KCL & KVL to write the Circuit Equation.

2. If Integrals, Differentiate to Differential Equation.

3. Assume a Solution form $K_1 + K_2 e^{st}$.

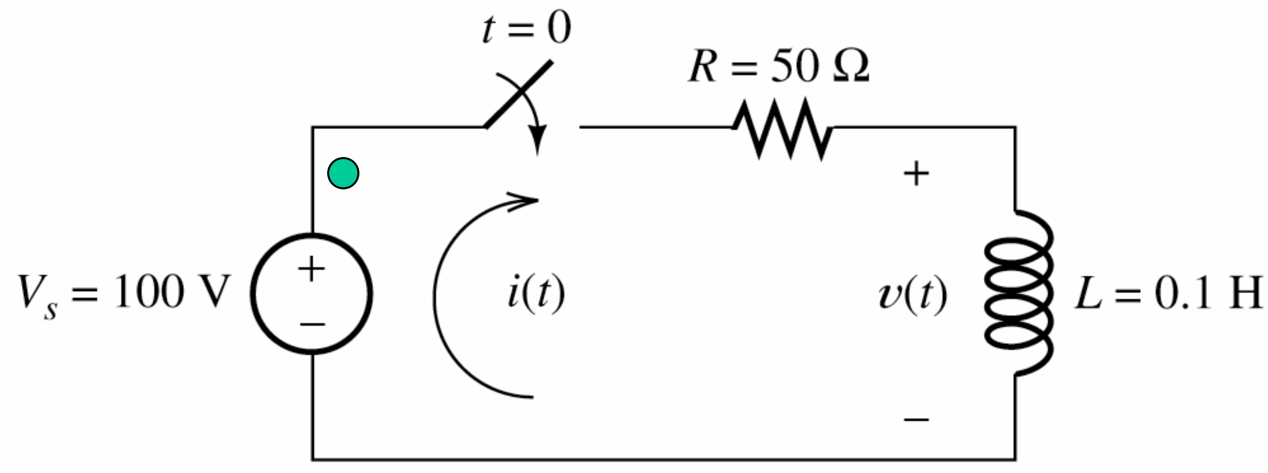
4. Substitute Solution into Differential equation to determine K_1 and s

5. Use the initial conditions to determine the value of K_2.

6. Write the final solution.

(Alternatively, we can determine ***K_1* by solving the circuit in steady state**)

RL Transient Analysis (Series & Switch On)



Initial Condition

$$i(t) = 0 \quad t < 0$$

Step 1. Apply KCL & KVL to write the circuit equation (No Step 2)

$$Ri(t) + L\frac{di(t)}{dt} = v_s$$

Step 3. Assume a solution of the form $K_1 + K_2e^{st}$.

$$i(t) = K_1 + K_2e^{st}$$

Step 4. Substitute the solution into the differential equation

$$\underbrace{RK_1}_{=v_s} + \underbrace{(RK_2 + sLK_2)}_{=\text{Zero}} e^{st} = v_s$$

$$K_1 = \frac{v_s}{R} \qquad s = \frac{-R}{L} \quad \Rightarrow \quad i(t) = v_s / R + K_2 e^{-tR/L}$$

Step 5. Use the initial conditions to determine the value of K_2.

$$i(0+) = v_s / R + K_2 = 0 \quad \Rightarrow \quad \boxed{i(t) = v_s / R(1 - e^{-tR/L})}$$

Time constant is $\tau = \dfrac{L}{R}$

Step 6. Write the final solution.

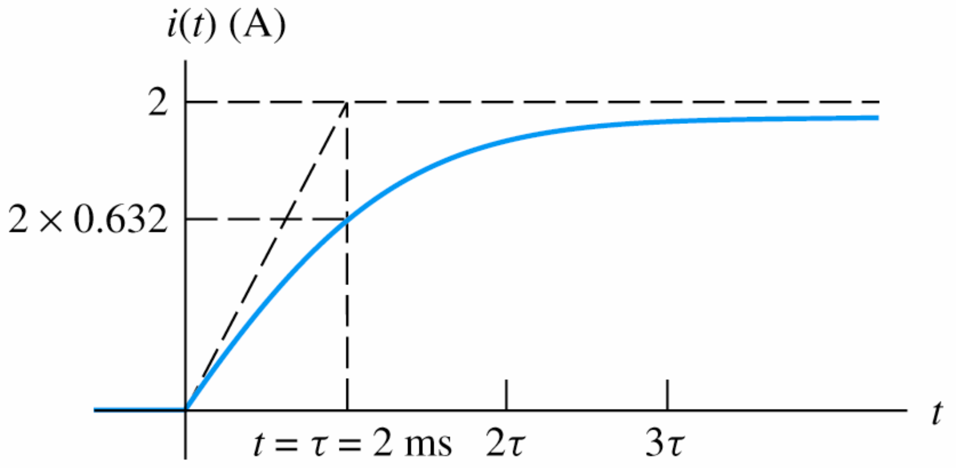
$$\frac{v_s}{R} = \frac{100}{50} = 2 \quad \tau = \frac{L}{R} = \frac{0.1}{50} = 2ms \quad \Rightarrow \quad i(t) = 2(1 - e^{-t/\tau})$$

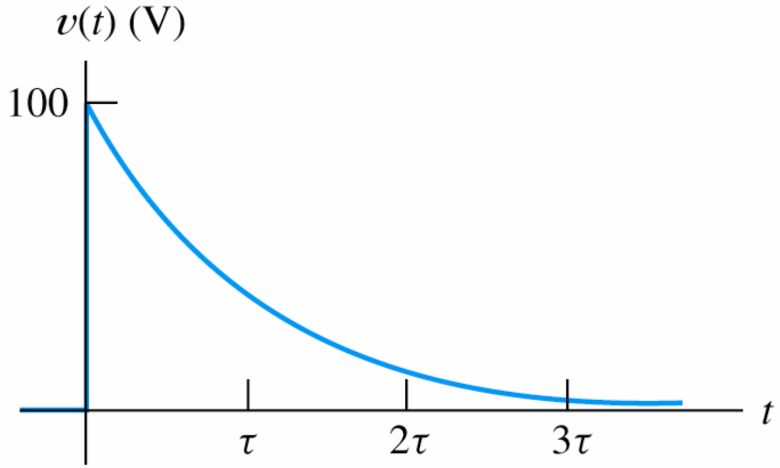
Using $\quad v_L(t) = L\dfrac{di_L(t)}{dt}$

$$i(t) = v_s / R(1 - e^{-tR/L}) \quad \Rightarrow \quad v_L(t) = v_s e^{-tR/L}$$

In given Circuit Condition, R=100Ω, L=0.1H

$$i(t) = 2(1 - e^{-t/\tau}) \qquad\qquad v_L(t) = 100e^{-t/\tau}$$





RL Transient Analysis (Parallel & Switch Off)

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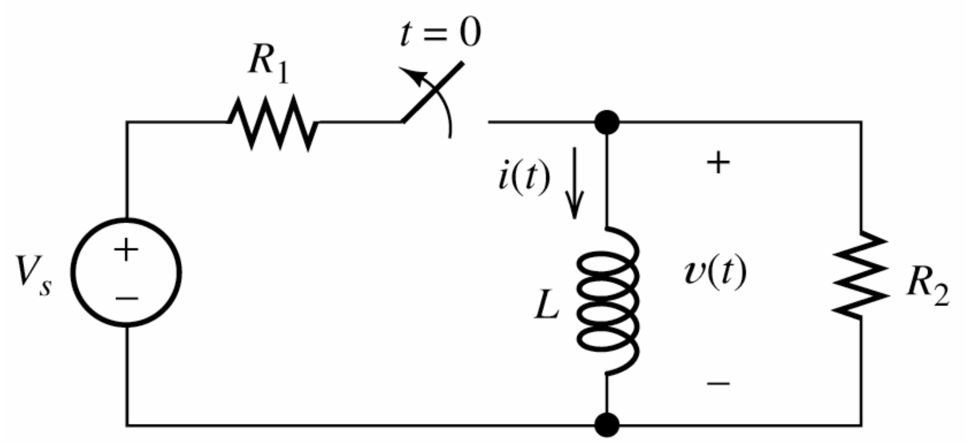

Initial Condition

$$i(t) = \frac{v_s}{R_1} \;\; \& \;\; v(t) = 0 \qquad t < 0$$

Before Switch Off, Current Circulates through R_1 , v_s & Inductor
After Switch Off, Current from v_s disappear
Current flows through inductor return to R_2

→ Voltage appears across R_2 & Inductor, causing to Current to decay
→ Steady-State Solution becomes Zero State

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Step 1. Apply KCL & KVL →Circuit equation (No Step 2)

$$Ri(t) + L\frac{di(t)}{dt} = v_s$$

Step 3&4. Assume Solution form $K_1 + K_2e^{st}$ $\quad i(t) = K_1 + K_2e^{st}$

Steady State is Zero State : $K_1 = 0$ $\quad i(t) = K_2e^{st}$

$$i(t) = K_2e^{-t/\tau} \qquad \tau = \frac{L}{R_2}$$

Step 5. Use the initial conditions to determine the value of K_2.

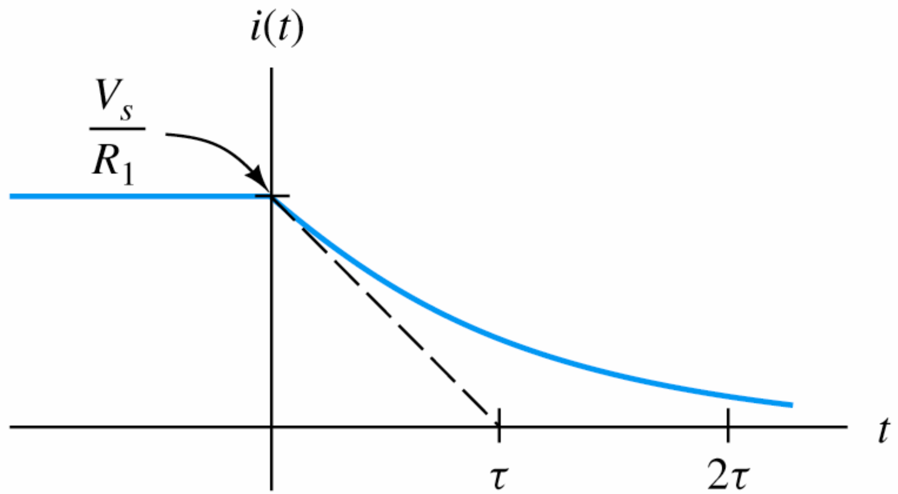
$$i(0+) = v_s / R_1 = K_2 \quad \Rightarrow \quad \boxed{i(t) = v_s / R_1 e^{-t/\tau}}$$

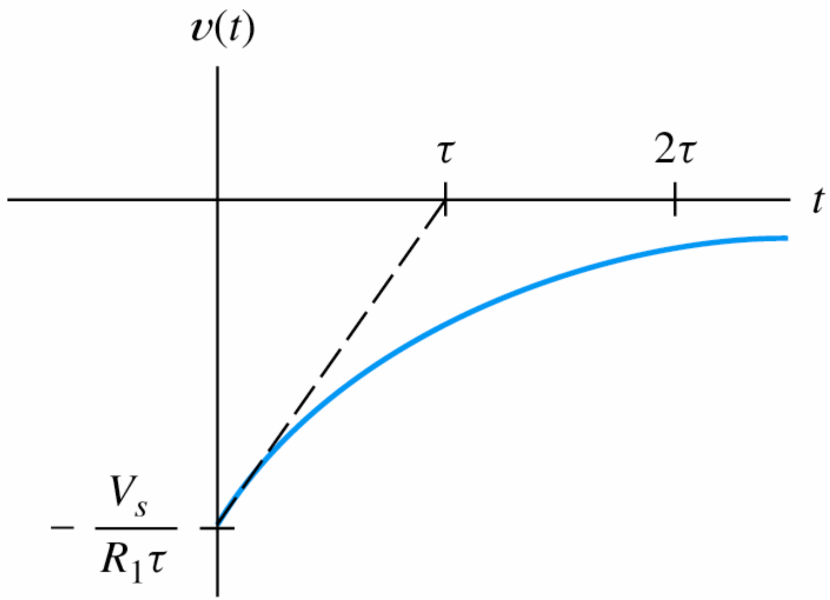
$$v(t) = 0 \quad t < 0$$

$$v_L(t) = L\frac{di_L(t)}{dt} \quad \Rightarrow \quad v(t) = -\frac{Lv_s}{R_1\tau}e^{-t/\tau} \quad t > 0$$

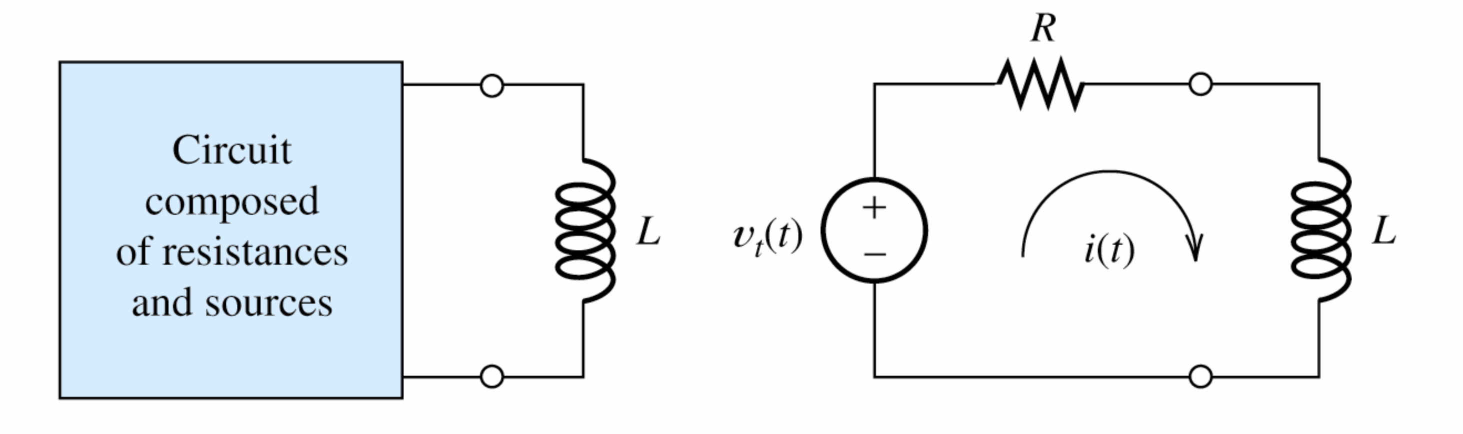
$$v(t) = 0 \quad t < 0$$

$$i(t) = v_s / R_1 e^{-t/\tau} \qquad v_L(t) = L\frac{di_L(t)}{dt} \Rightarrow v(t) = -\frac{Lv_s}{R_1\tau}e^{-t/\tau} \quad t > 0$$





RC & *RL* Circuits with General Sources



When Inductor is Linked with General Source

→ Using Thévenin Theorem → Voltage Source + Resistor

Note : *Voltage Source & Current are varying with time*

KVL

$$L\frac{di_L(t)}{dt} + Ri(t) = v_t(t) \quad \Longrightarrow \quad \frac{L}{R}\frac{di_L(t)}{dt} + i(t) = \frac{v_t(t)}{R}$$

Generally $$\tau\frac{dx(t)}{dt} + x(t) = f(t)$$

τ : Time Constant

f(t) : Forcing function

Solution of 1st Order Differential Equation

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$$y' + p(t)y = r(t) \xrightarrow[\textit{Appendix}]{} y = \underline{e^{-\int p(t)dt}\int e^{\int p(t)dt} r(t)dt} + \underline{C\ e^{-\int p(t)dt}}$$

The particular solution (called the forced response) is any expression that satisfies the equation.

The complementary solution (called the natural response) is obtained by solving the homogeneous equation.

$$\tau\frac{dx(t)}{dt} + x(t) = f(t) \Longrightarrow x(t) = x_p(t) + x_c(t)$$

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = \frac{1}{\tau}f(t) \Longrightarrow p(t) = \frac{1}{\tau} \quad \& \quad r(t) = \frac{1}{\tau}f(t)$$

$$y = e^{-\int \frac{1}{\tau}dt}\int e^{\int \frac{1}{\tau}dt}\frac{1}{\tau}f(t)dt + C\ e^{-\int \frac{1}{\tau}dt} = e^{-\frac{t}{\tau}}\int e^{\frac{t}{\tau}}\frac{1}{\tau}f(t)dt + C\ e^{-\frac{t}{\tau}}$$

$$\frac{dx_c(t)}{dt} + \frac{1}{\tau}x_c(t) = 0 \qquad \frac{dx_c(t)/dt}{x_c(t)} = -\frac{1}{\tau} \qquad \ln x_c(t) = -\frac{t}{\tau} + c \qquad x_c(t) = Ke^{-\frac{t}{\tau}}$$

Step-by-Step Solution

Circuits with a resistance, a source, and an inductance (or a capacitance)

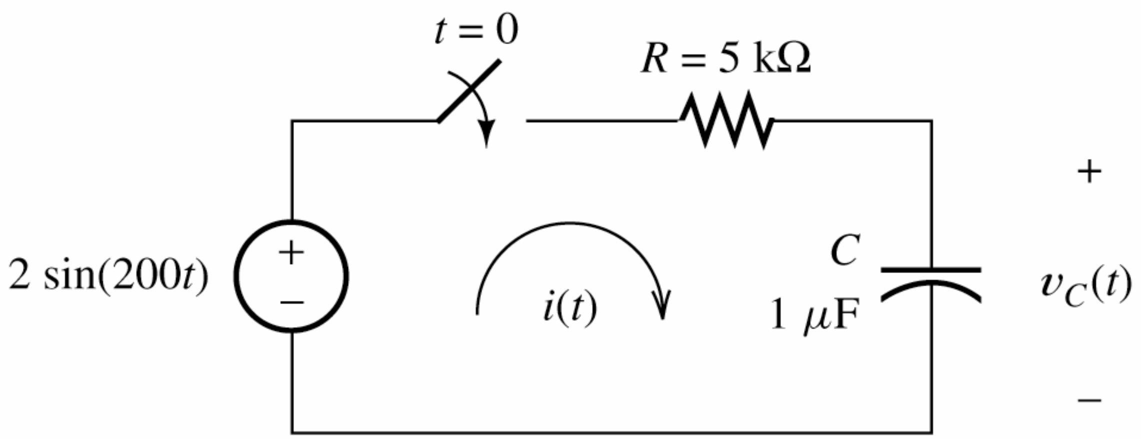
1. Write *Circuit Equation* as *1st Order* Differential Equation.

2. Find a *Particular Solution*.

The details of this step depend on the form of the forcing function.

3. Obtain the *Complete Solution* by adding the particular solution to the *Complementary Solution*

4. Use *Initial Conditions* to find the value of *K*.



Use KVL

$v_C(0) = 1$ V

$$Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_c(0) - 2\sin(200t) = 0$$

Step 1. Write Circuit Equation as 1st-Order Differential Equation.

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 400\cos(200t) \Longrightarrow RC\frac{di(t)}{dt} + i(t) = 400C\cos(200t)$$

Step 2. Find a particular solution.

Using General Solution or Guessing from Math. $\Longrightarrow$ $i_p(t) = A\cos(200t) + B\sin(200t)$

$$5\times10^{-3}\frac{di(t)}{dt} + i(t) = 400\times10^{-6}\cos(200t) \Longrightarrow i_p(t) = 200\cos(200t) + 200\sin(200t)\,\mu A$$

$A=B=200\mu A$

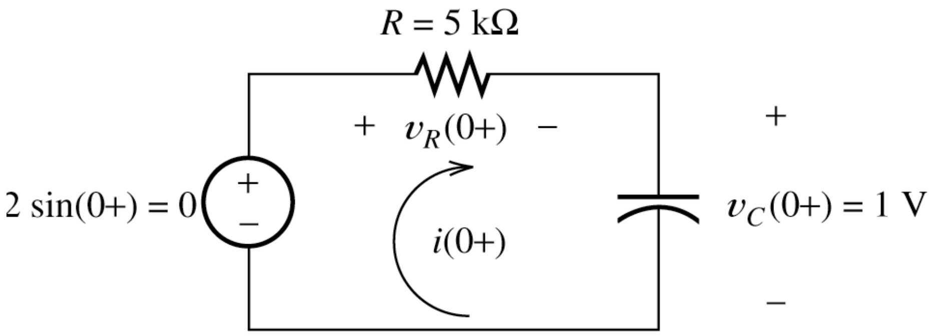
Step 3. Obtain the Complete Solution

$$RC\frac{di(t)}{dt}+i(t)=0 \Longrightarrow i_c(t)=Ke^{-\frac{t}{RC}}=Ke^{-\frac{t}{\tau}}$$

$$i(t)=200\cos(200t)+200\sin(200t)\mu A+Ke^{-\frac{t}{RC}}$$

Step 4. Use initial conditions to find the value of *K*.

$$i(0+)=\frac{v_R(0+)}{R}=\frac{-1}{5000}=-200\mu A=(200+K)\mu A$$

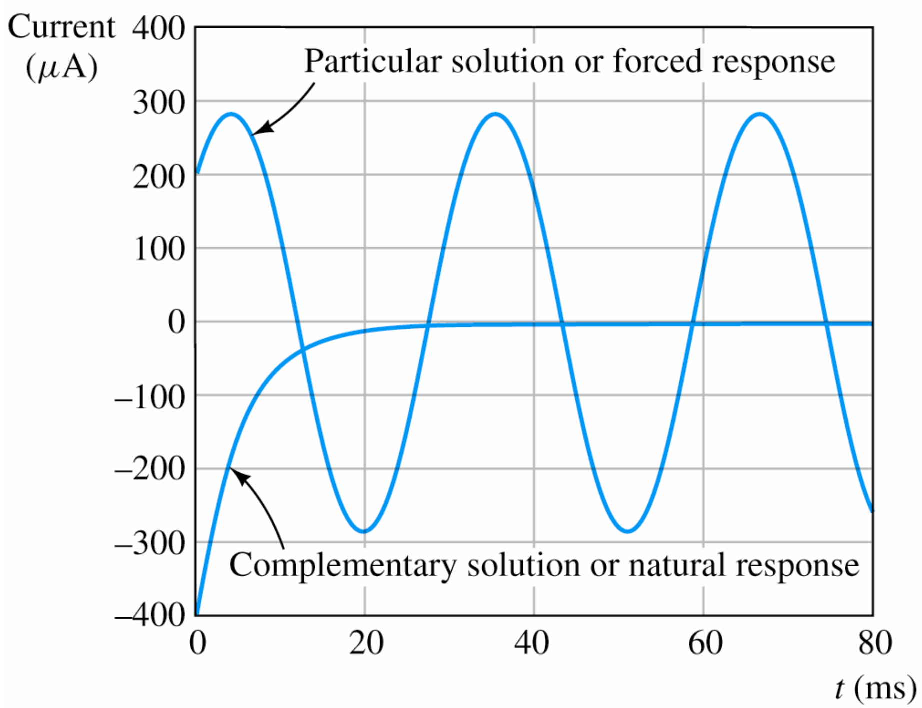


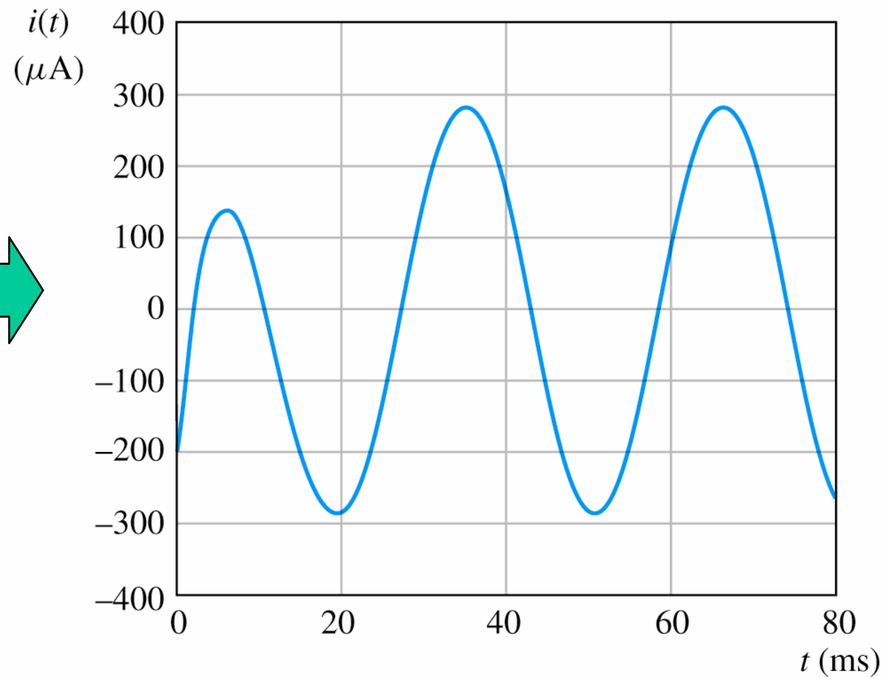
$$K=-400\mu A$$

$$i(t)=200\cos(200t)+200\sin(200t)\mu A-400e^{-\frac{t}{RC}}\quad \mu A$$

$$\tau=RC=25ms$$

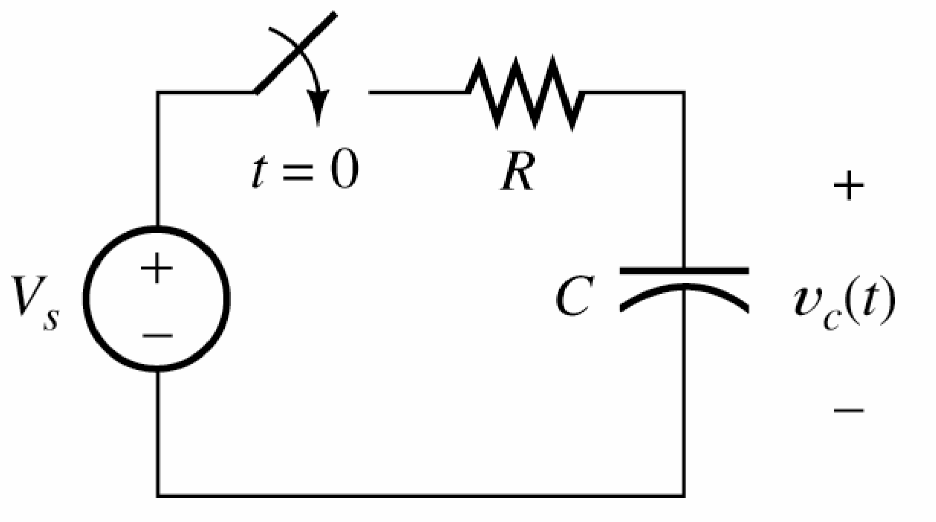
$$i(t) = 200\cos(200t) + 200\sin(200t)\,\mu A - 400e^{-\frac{t}{RC}}\ \ \mu A$$

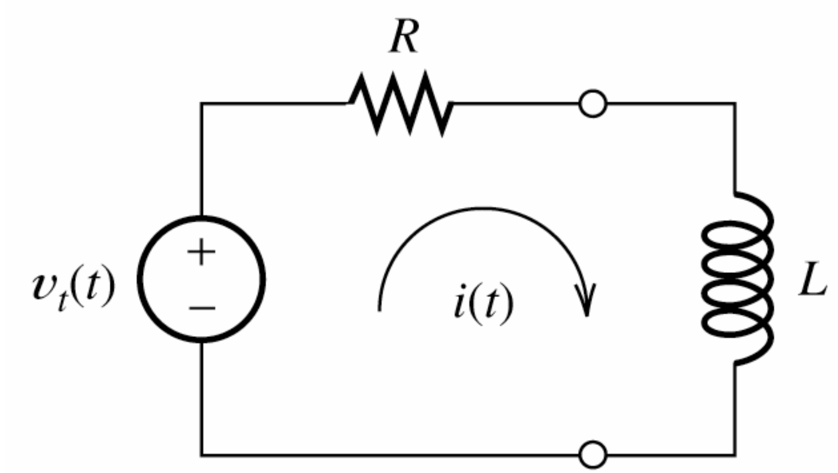




Summary of 1st Order Diff. Equation

$$i(t) = e^{-\frac{t}{\tau}} \int e^{\frac{t}{\tau}} \frac{1}{\tau} f(t)dt + C\ e^{-\frac{t}{\tau}}$$





Constant Voltage Source (DC)

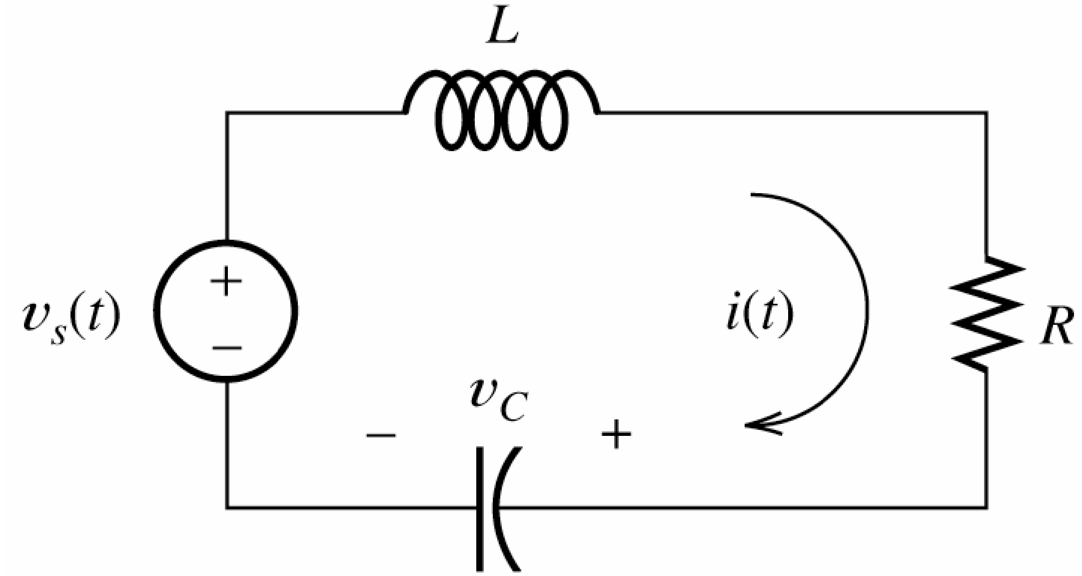
$$i(t) = e^{-\frac{t}{\tau}} \int e^{\frac{t}{\tau}} \frac{1}{\tau} K dt + C\ e^{-\frac{t}{\tau}}$$

$$= K_1 + K_2 e^{st}$$

General Voltage Source

$$i(t) = e^{-\frac{t}{\tau}} \int e^{\frac{t}{\tau}} \frac{1}{\tau} f(t)dt + C\ e^{-\frac{t}{\tau}}$$

Second-Order Circuits



Using KVL

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(t)dt+v_C(0)=v_s(t)$$

Taking Derivative & Divide by L

$$\frac{d^2i(t)}{dt^2}+\frac{R}{L}\frac{di(t)}{dt}+\frac{1}{LC}i(t)=\frac{1}{L}\frac{dv_s(t)}{dt}$$

$$\alpha = \frac{R}{2L}$$: Damping Coefficient

$$\omega_0 = \frac{1}{\sqrt{LC}}$$: Undamped Resonant Frequency

$$f(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$: Forcing function

$$\boxed{\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)}$$

➡ Linear 2nd Order Differential Equation

Here We know that $x(t) = x_p(t) + x_c(t)$

$$x_p(t) : \frac{d^2 x_p(t)}{dt^2} + 2\alpha \frac{dx_p(t)}{dt} + \omega_0^2 x_p(t) = f(t)$$: Particular Solution

$$x_c(t) : \frac{d^2 x_c(t)}{dt^2} + 2\alpha \frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$: Complementary Solution

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$\boxed{x_p(t)}$: Particular Solution : $\frac{d^2x_p(t)}{dt^2}+2\alpha\frac{dx_p(t)}{dt}+\omega_0^2x_p(t)=f(t)$

In Electric Circuit, forcing function is Primarily DC & Sinusoidal Sources

<u>DC Source</u> : Steady State Solution

Can be obtained by Replacing Inductor to Short Circuit & Capacitor to Open Circuit

<u>Sinusoidal Source</u> : Complex Number Analysis (Chapter 5)

$\boxed{x_c(t)}$: Complementary Solution : $\frac{d^2x_c(t)}{dt^2}+2\alpha\frac{dx_c(t)}{dt}+\omega_0^2x_c(t)=0$

Let $x_c(t)=Ke^{st}$ $\Longrightarrow$ $(s^2+2\alpha s+\omega_o^2)Ke^{st}=0$

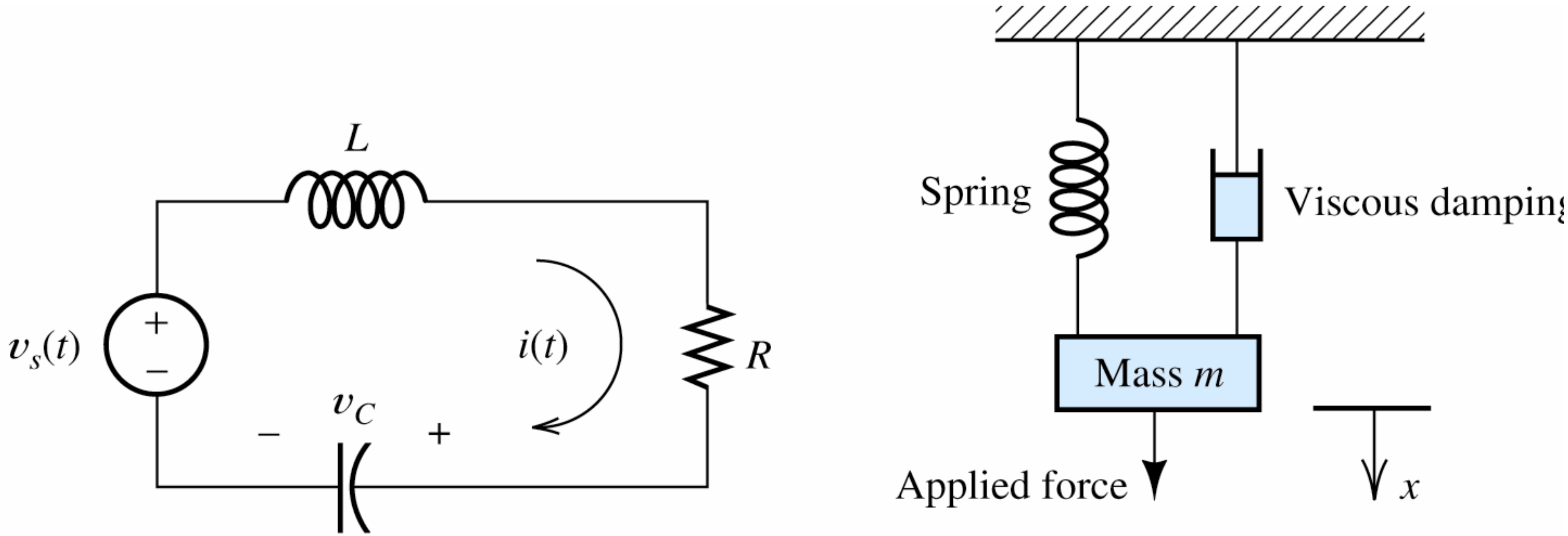
Generally $s^2+2\alpha s+\omega_o^2=0$: Characteristic Equation

Two Solutions : $s_1=-\alpha+\sqrt{\alpha^2-\omega_0^2}$

$s_2=-\alpha-\sqrt{\alpha^2-\omega_0^2}$

$\zeta=\frac{\alpha}{\omega_0}$: Damping Ratio

2^{nd} Order Circuit Analogy



(a) Electrical circuit

(b) Mechanical analog

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t)dt = \frac{1}{L}\frac{dv_s(t)}{dt}$$

$$\frac{d^2 x(t)}{dt^2} + \frac{C}{m}\frac{dx(t)}{dt} + \frac{k}{m} x(t) = \frac{1}{m} f(t)$$

Solution Case of 2nd Order Equation

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1. ***<u>Overdamped case</u>*** : $\zeta > 1$ or $\alpha > \omega_0$: Real & Distinct Two Roots

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

2. ***<u>Critically damped case</u>*** : $\zeta = 1$ $\alpha = \omega_0$: Real & Equal Roots.

$$x_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$

3. ***<u>Underdamped case</u>*** : $\zeta < 1$ or $\alpha < \omega_0$: Complex Roots

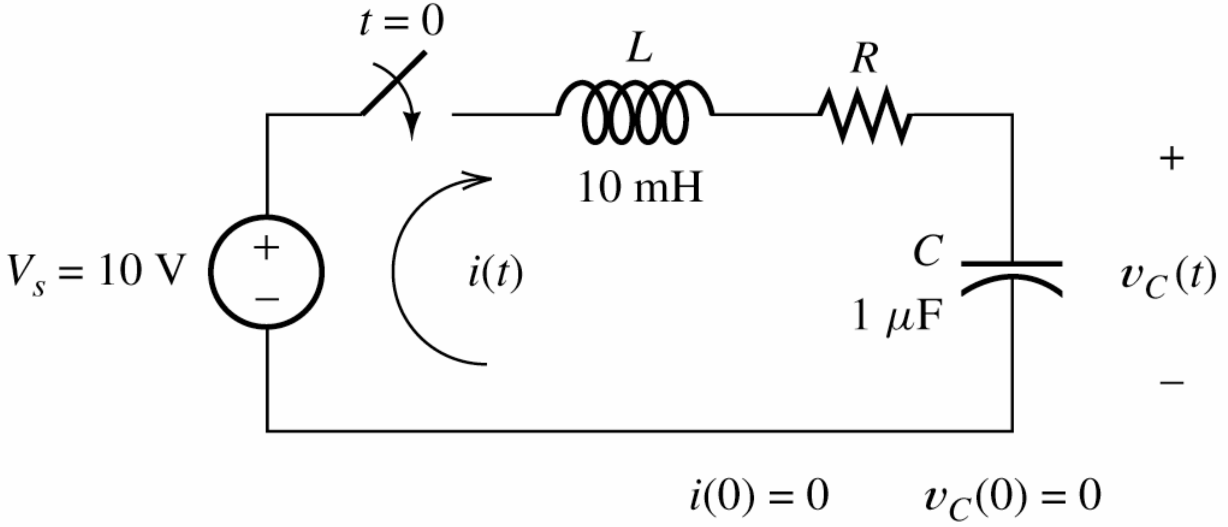
$$s_1 = -\alpha + j\omega_n \ \& \ s_2 = -\alpha - j\omega_n$$

Natural frequency is given by $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$

$$x_c(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)$$

In electrical engineering, we use *j* rather than *i* to stand for square root of -1, because of *i* for current.

Analysis of 2nd Order Circuit with DC Source



$$\text{KVL}: L\frac{di(t)}{dt} + Ri(t) + v_C(t) = V_s \Leftarrow i(t) = C\frac{dv_c(t)}{dt}$$

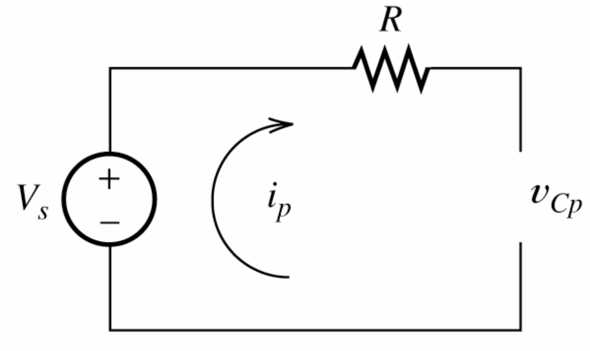
$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_C(t) = V_s \Rightarrow \frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC}$$

$$\frac{d^2v_c(t)}{dt^2} + 2\alpha\frac{dv_c(t)}{dt} + \omega_o^2 v_C(t) = \frac{V_s}{LC}$$

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Particular Solution



Set Inductor as Short Circuit
Capacitor as Open Circuit

$$\Rightarrow \quad v_{cp}(t) = V_s$$

Complementary Solution

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^4$$

R=300Ω $\alpha = 1.5\times10^4$ & $\zeta = 1.5$ $s_1 = -2.618\times10^4$ $s_2 = -0.382\times10^4$

$$v_c(t) = 10 + 1.708e^{s_1 t} - 11.708e^{s_2 t}$$

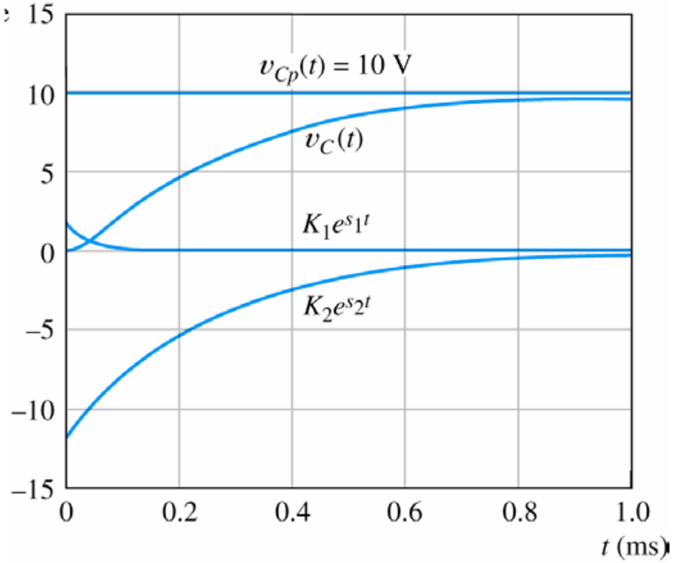
R=200Ω $\alpha = 10^4$ & $\zeta = 1$ $s_1 = s_2 = -10^4$

$$v_c(t) = 10 - 10e^{s_1 t} - 10^5 te^{s_2 t}$$

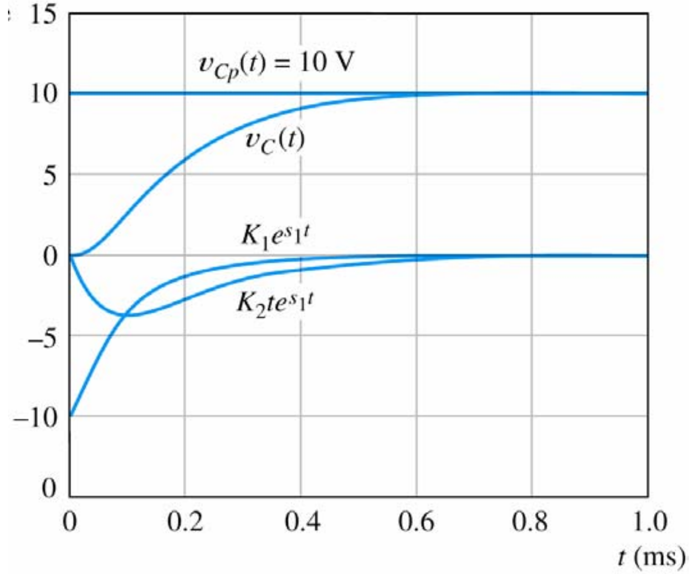
R=100Ω $\alpha = 0.5\times10^4$ & $\zeta = 0.5$ $\omega_n = 8660$

$$v_c(t) = 10 - 10e^{-\alpha t}\cos(\omega_n t) - 5.774e^{-\alpha t}\sin(\omega_n t)$$

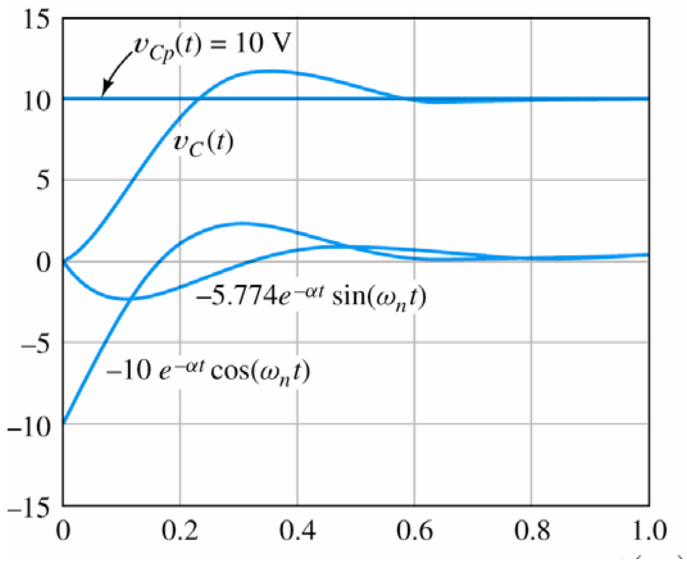
R=300Ω

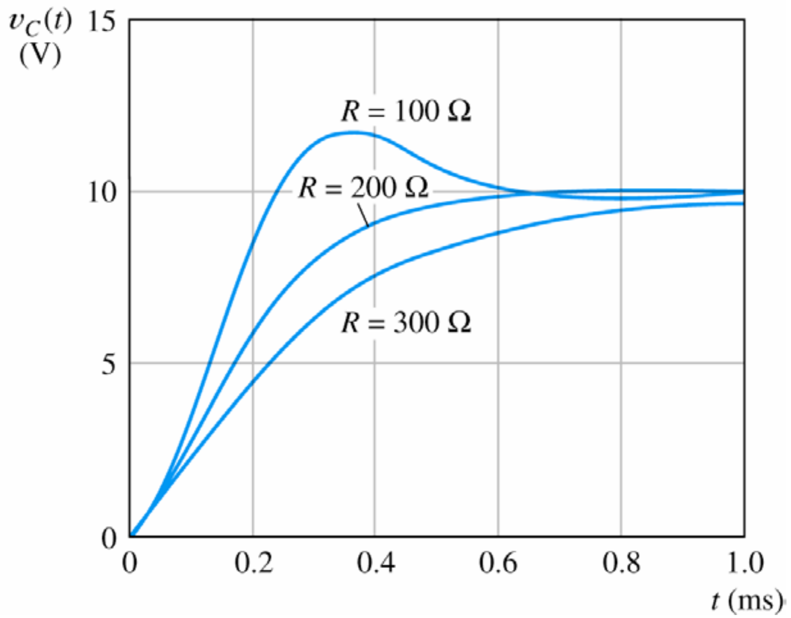


R=200Ω

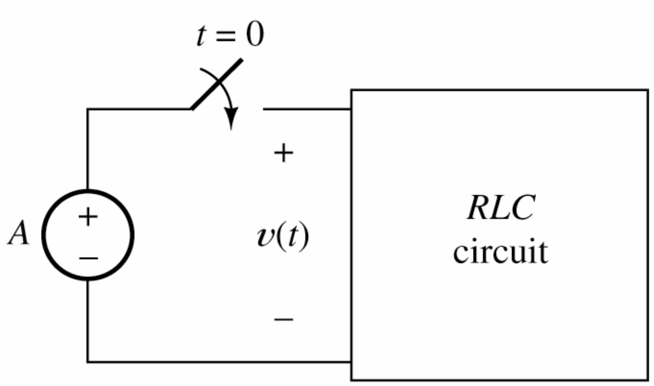


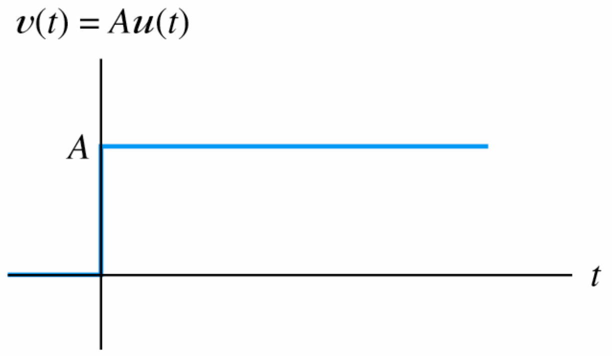
R=100Ω

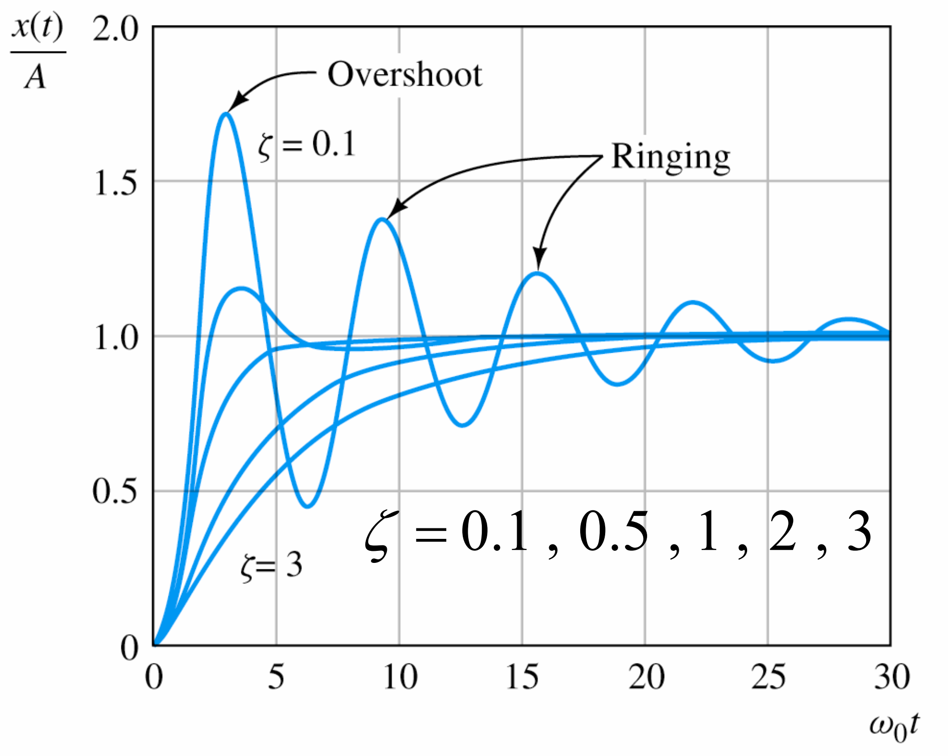




General RLC Circuit with DC Source

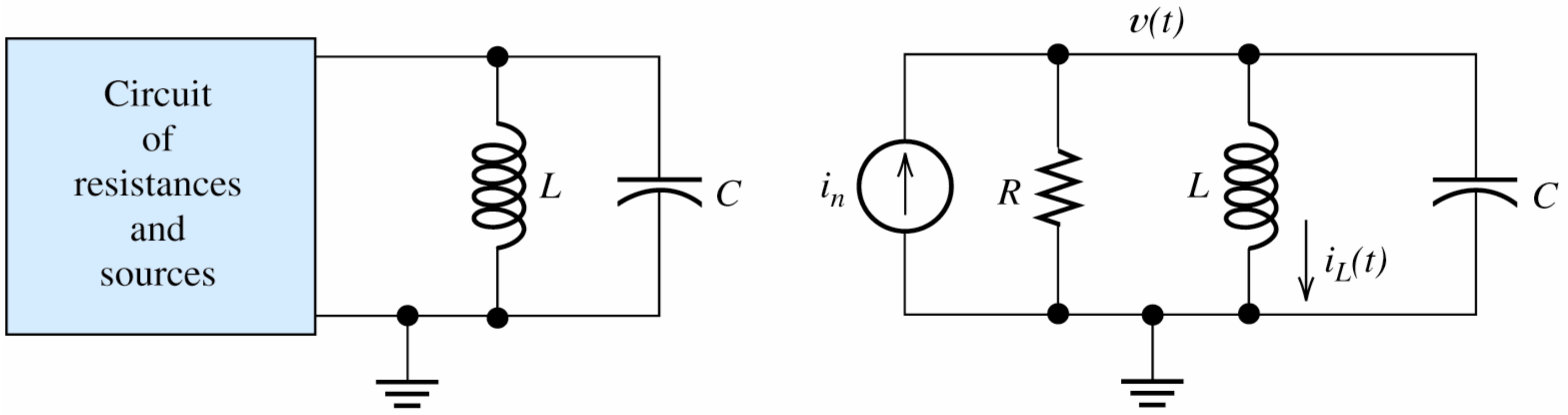






To avoid Overshoot,
Design Damping Ratio to 1

RLC Circuit with Parallel L & C



Using KCL

$$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)+\frac{1}{L}\int_0^t v(t)dt+i_L(0)=i_n(t)$$

Taking Derivative & Divide by C

$$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=\frac{1}{C}\frac{di_n(t)}{dt}$$

SNUMSE
K.H. Oh

$\alpha = \frac{1}{RC}$: Damping Coefficient

$\omega_0 = \frac{1}{\sqrt{LC}}$: Undamped Resonant Frequency

$f(t) = \frac{1}{C}\frac{di_n(t)}{dt}$: Forcing function

$$\boxed{\frac{d^2v(t)}{dt^2} + 2\alpha\frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t)}$$

➡ Linear 2nd Order Differential Equation

$$x(t) = x_p(t) + x_c(t)$$

All treatments are same as Serial !